Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 231E – Engineering Calculus

Midterm 5 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Tutors: Nov. 8, 2-3:50pm Yash and Aditya

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/979
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Test the following improper integral for convergence. If it converges, find the value

$$\int_0^\infty e^{-\frac{4}{3}y}dy$$

(a) Diverges
(b) Converges, value is $-\frac{3}{4}$
(c) Converges, value is $\frac{4}{3}$
(d) Converges, value is $\frac{3}{4}$
(e) None of the above

This is an improper integral, so we need to check for convergence.

$$\lim_{t\to\infty}\int_0^t e^{-\frac{4}{3}y}dy$$

$$\lim_{t\to\infty}\left(\frac{-3}{4}e^{\frac{-4}{3}y}\right)\Big|_0^t$$

$$\lim_{t\to\infty}\left(\left(\frac{-3}{4}e^{-\frac{4}{3}t}\right)-\left(-\frac{3}{4}e^{0}\right)\right)$$

$$\lim_{t\to\infty}\frac{-3}{4e^{\frac{4}{3}t}}+\frac{3}{4}$$

$$\frac{-3}{\infty}+\frac{3}{4}=\boxed{\frac{3}{4}}$$

2. For which value of C does the following series converge?

$$\sum_{n=3}^{\infty}\left(\frac{C}{n+2}-\frac{n}{3n^2+1}\right)$$

(a) It does not converge for any value of C.

(b) $\frac{1}{2}$

(c) 1

(d) $\frac{1}{3}$

(e) 0

$$\frac{C}{n+2}-\frac{n}{3n^2+1}=\frac{C(3n^2+1)}{(n+2)(3n^2+1)}-\frac{n(n+2)}{(n+2)(3n^2+1)}$$

$$\frac{3Cn^2+C-n^2-2n}{(n+2)(3n^2+1)}=\frac{(3C-1)n^2-2n+C}{3n^3+6n^2+n+2}$$

$$\sum_{n=3}^{\infty}\frac{(3C-1)n^2-2n+C}{3n^3+6n^2+n+2}$$

If $\boxed{C=\frac{1}{3}}$, then the limit comparison test with $\sum_{n=3}^{\infty}\frac{1}{n^2}$ shows that the series converges

If C does not equal $\frac{1}{3}$, then the limit comparison test with $\sum_{n=3}^{\infty}\frac{1}{n}$ shows that the series diverges

3. If $a_n = 3 + \frac{\ln(n)^6}{\sqrt{n}}$, then the sequence a_n:

(a) Converges to 3.

(b) Diverges to infinity.

(c) Diverges, but not to infinity.

(d) Converges to 6.

(e) Converges to 4.

$$a_n = 3 + \frac{\ln(n)^6}{\sqrt{n}}$$

$$\lim_{n\to\infty} 3 + \frac{\ln(n)^6}{\sqrt{n}}$$

For large n, $\ln(n) < n^c < c^n$

$$\frac{\ln(n)^6}{\sqrt{n}} < \frac{(n^{\frac{1}{24}})^6}{n^{\frac{1}{2}}} = \frac{1}{n^{\frac{1}{4}}}$$

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{4}}} = 0$$

Therefore,

$$\lim_{n\to\infty} \frac{\ln(n)^6}{\sqrt{n}}$$

must also converge to zero.

$$3 + \lim_{n\to\infty} \frac{\ln(n)^6}{\sqrt{n}} = 3 + 0 = \boxed{3}$$

4. Determine if the following integral converges or diverges

$$\int_1^\infty \frac{x^3+12x-2}{x^6+5x^5+3x^2-1}dx$$

You can use the limit comparison test to prove the behavior of the improper integral

$$f(x)=\frac{x^3+12x-2}{x^6+5x^5+3x^2-1}dx$$

The denominator is bottom heavy by three more powers of x so we expect it to converge

$$g(x)=\frac{1}{x^3}$$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^3+12x-2}{x^6+5x^5+3x^2-1}*\frac{x^3}{1}$$

$$\lim_{x\to\infty}\frac{x^6+12x^4-2x^3}{x^6+5x^5+3x^2-1}$$

$$\lim_{x\to\infty}\frac{x^6+12x^4-2x^3}{x^6+5x^5+3x^2-1}*\frac{\frac{1}{x^6}}{\frac{1}{x^6}}$$

$$\lim_{x\to\infty}\frac{1+\frac{12}{x^2}-\frac{2}{x^3}}{1+\frac{5}{x}+\frac{3}{x^4}-\frac{1}{x^6}}=\frac{1}{1}=1$$

$\int_1^\infty g(x)$ converges by p-test, therefore $\int_1^\infty f(x)$ also $\boxed{\text{converges}}$ by Limit Comparison Test

5. Determine which of the following is true about the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

(a) Not enough information to make a determination

(b) The series converges to $\frac{2}{5}$

(c) The series diverges

(d) The series converges absolutely

(e) The series converges only conditionally

The correct answer for this problem is (d). $\frac{1}{n^2}$ converges by the p-test. Since a_n is decreasing and $a_n \to 0$ as $n \to \infty$, the series absolutely converges.

6. Find the sum of the geometric series:

$$12 + 12\left(\frac{7}{19}\right) + 12\left(\frac{7}{19}\right)^2 + 12\left(\frac{7}{19}\right)^3 \ldots$$

(a) 20

(b) 8

(c) 19

(d) 14

(e) $\frac{7}{19}$

This is a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} |r| < 1$$

$r = \frac{7}{19}$, $a = 12 \rightarrow \frac{12}{1-\frac{7}{19}} = \boxed{19}$

7. Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{\cos^n(x)}{3^n}$$

(a) $\frac{3}{3-\cos(x)}$

(b) $\frac{1}{3-\cos(x)}$

(c) $\frac{2}{3-\cos(x)}$

(d) $\frac{3}{3-\sin(x)}$

(e) $\frac{1}{3-\sin(x)}$

Identify the geometric series

$$\sum_{n=0}^{\infty} ar^{n-1}, \; r = \frac{(n+1)_{term}}{n_{term}} = \left[\frac{\cos^{n+1}(x)}{3^{n+1}}\right] \frac{3^n}{\cos^n(x)}$$

$$\frac{\cos^1(x)}{3} = \frac{1}{3}\cos(x) < 1$$

This converges to: $\frac{a}{1-r}$

$$a = 1, r = \frac{\cos(x)}{3} \rightarrow \boxed{\frac{3}{3-\cos(x)}}$$

8. Determine the value of the series, if it converges:

$$\sum_{n=1}^{\infty} \frac{5^{n+2}}{6^n}$$

(a) 64

(b) The series diverges

(c) 2

(d) 125

(e) 25

$$\sum_{n=1}^{\infty} \frac{5^{n+2}}{6^n} \to \left(\frac{5}{6}\right)5^2 \qquad r = \frac{5}{6}$$

$\frac{a}{1-r}$ where a is the first term.

$$a = \frac{5^3}{6}$$

$$\frac{5^3}{6}\left(\frac{1}{1-\frac{5}{6}}\right) = 5^3 = \boxed{125}$$

9. Does the series converge or diverge?

$$\sum_{n=1}^{\infty} e^{\frac{3}{n}}$$

Test for divergence

$$\lim_{n\to\infty} e^{\frac{3}{n}} \to 1$$

It $\boxed{\text{diverges}}$.

10. Determine whether the sequence

$$a_n = \left(\frac{n+1}{n+3}\right)^{2n}$$

converges or diverges. If it converges, find its limit.

(a) 0

(b) e^{-2}

(c) e^{-4}

(d) 1

(e) The sequence diverges.

Take the natural log:

$$\ln(a_n) = 2n \ln\left(\frac{n+1}{n+3}\right).$$

Using l'Hôpital's rule,

$$\lim_{x\to\infty} \frac{2\ln\left(\frac{x+1}{x+3}\right)}{1/x} = \lim_{x\to\infty} \frac{-4x^2}{(x+1)(x+3)} = -4.$$

Therefore,

$$\lim_{n\to\infty} a_n = \boxed{e^{-4}}.$$

The answer is **(c)**.

11. Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n+1}$$

converges absolutely, converges conditionally, or diverges.

(a) It converges absolutely by the ratio test.

(b) It converges conditionally.

(c) It diverges because its terms do not approach zero.

(d) It converges absolutely by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.

(e) It diverges because the magnitudes of its terms increase.

Use the alternating series test.

$$b_n = \frac{\sqrt{n}}{n+1} \to 0.$$

To check that the terms decrease,

$$f(x) = \frac{\sqrt{x}}{x+1}, \qquad f'(x) = \frac{1-x}{2\sqrt{x}(x+1)^2} \le 0 \quad \text{for } x \ge 1.$$

So the alternating series converges.

For absolute convergence, compare with $\sum 1/\sqrt{n}$:

$$\lim_{n\to\infty}\frac{\sqrt{n}/(n+1)}{1/\sqrt{n}} = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

Since $\sum 1/\sqrt{n}$ diverges by the p-test, the original series converges conditionally. The answer is **(b)**.

12. Use the ratio test to analyze the convergence of

$$\sum_{n=1}^{\infty}(-1)^n\frac{3^n n!}{n^n}.$$

Which option gives the correct value of

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and the corresponding conclusion?

(a) $L=0$; the series converges absolutely.

(b) $L=1$; the test is inconclusive.

(c) $L=e/3$; the series converges absolutely.

(d) $L=3/e$; the series diverges.

(e) $L=1/3$; the series converges absolutely.

Using the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}\frac{n^n}{3^n n!}=3\left(\frac{n}{n+1}\right)^n.$$

$$L=\lim_{n\to\infty}\frac{3}{(1+1/n)^n}=\frac{3}{e}>1.$$

Therefore, the series $\boxed{\text{diverges}}$. The answer is **(d)**.

13. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\,3^n}.$$

(a) $R = 3, \quad I = [-1, 5)$

(b) $R = 3, \quad I = (-1, 5]$

(c) $R = 3, \quad I = (-1, 5)$

(d) $R = 3, \quad I = [-1, 5]$

(e) $R = 1, \quad I = (1, 3)$

Use the ratio test:

$$L = \lim_{n\to\infty} \frac{|x-2|}{3}\frac{n}{n+1} = \frac{|x-2|}{3}.$$

$$\frac{|x-2|}{3} < 1 \quad \Longrightarrow \quad -1 < x < 5.$$

Check the endpoints:

$$x = -1: \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{converges by the alternating series test.}$$

$$x = 5: \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges.}$$

Therefore, $\boxed{R = 3, \quad I = [-1, 5)}$. The answer is **(a)**.

14. Find a power series representation for

$$f(x) = \frac{1}{3-x}$$

centered at $x = 0$.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$

(d) $\displaystyle\sum_{n=0}^{\infty} 3^n x^n$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{3^{n+1}}$

Use the geometric series:

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \qquad |r| < 1.$$

$$\frac{1}{3-x} = \frac{1}{3}\frac{1}{1-x/3} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \boxed{\sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}}.$$

This is valid for $|x| < 3$. The answer is **(c)**.

15. By differentiating a geometric series, find a power series representation for

$$f(x) = \frac{1}{(2-x)^2}$$

centered at $x = 0$.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^{n+1}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^{n+2}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^{n+2}}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)x^n}{2^{n+2}}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{2^{n+2}}$

Start with the geometric series:

$$\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-x/2} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}.$$

Differentiate both sides:

$$\frac{1}{(2-x)^2} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}}.$$

Change the index so the sum starts at $n = 0$:

$$\frac{1}{(2-x)^2} = \boxed{\sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^{n+2}}}.$$

This is valid for $|x| < 2$. The answer is **(b)**.

16. By integrating a geometric series, find a power series representation for

$$f(x) = \ln(1 + 2x)$$

centered at $x = 0$.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}(2x)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2x)^n}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n+1}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n}$

Start with

$$\frac{2}{1+2x} = 2\sum_{n=0}^{\infty}(-2x)^n = \sum_{n=0}^{\infty}(-1)^n 2^{n+1}x^n.$$

Integrate both sides:

$$\ln(1+2x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}x^{n+1}}{n+1}.$$

At $x = 0$, $\ln 1 = 0$, so $C = 0$. Changing the index gives

$$\ln(1+2x) = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n}}.$$

This is valid for $|x| < 1/2$. The answer is **(e)**.