Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 231E – Engineering Calculus

Midterm 4 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Tutors: Oct. 27, 3-4:50pm Yash, Aditya, Aadya

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/979
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Which of the following integrals best represents the length of the curve $y = \cos(x)$ between $x = 0$ and $x = 1$?

(a) $\displaystyle\int_0^1 \sqrt{1 + \cos^2(x)}\, dx$

(b) $\displaystyle\int_0^1 \sqrt{1 + \sin^2(x)}\, dx$

(c) $\displaystyle\int_0^1 x\sqrt{1 + \cos^2(x)}\, dx$

(d) $\displaystyle\int_0^1 x \sin(x)\, dx$

(e) $\displaystyle\int_0^1 x\sqrt{1 + \sin^2(x)}\, dx$

The formula for arc length is

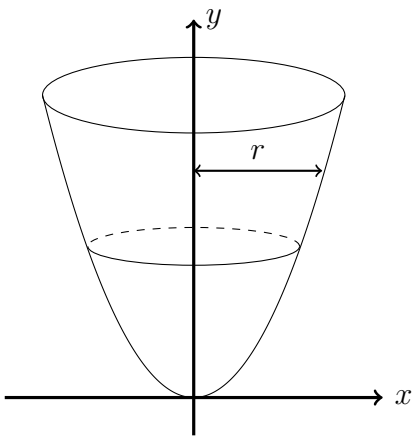
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = -\sin(x),$$

$$\boxed{L = \int_0^1 \sqrt{1 + \sin^2(x)}\, dx}$$

Answer: **(b)**.

2. The figure below represents a solid of revolution formed by rotating $y = x^2$ about the y-axis. Which integral represents the volume of the solid between $y = 0$ and $y = 1$?



(a) $\displaystyle\int_0^1 \pi y^2\,dx$

(b) $\displaystyle\int_0^1 \pi x^2\,dy$

(c) $\displaystyle\int_0^1 2\pi xy\,dx$

(d) $\displaystyle\int_0^1 2\pi xy\,dy$

(e) None of the above

Using the disk method,

$$V = \int_a^b \pi r^2\,dy$$

The radius is $r = x = \sqrt{y}$.

Therefore,

$$\boxed{V = \int_0^1 \pi x^2\,dy}$$

Answer: **(b)**.

3. Compute the arc length of the function

$$y = 1 + 2x^{3/2}$$

between $x = 0$ and $x = 1$.

(a) $\dfrac{14}{9}$

(b) $\dfrac{10}{9}$

(c) $\dfrac{2\sqrt{10}}{9}$

(d) $\dfrac{2}{27}(10\sqrt{10}+1)$

(e) None of the above

The arc length formula is

$$L = \int_0^1 \sqrt{1+(y')^2}\,dx$$

$$y' = 3\sqrt{x}$$

Thus,

$$L = \int_0^1 \sqrt{1+9x}\,dx$$

Let $u = 1 + 9x$, so $du = 9\,dx$.

The new bounds are $u = 1$ and $u = 10$.

$$L = \frac{1}{9}\int_1^{10} u^{1/2}\,du$$

$$L = \frac{2}{27}\left[u^{3/2}\right]_1^{10}$$

$$\boxed{L = \frac{2}{27}(10\sqrt{10}-1)}$$

Answer: **(e)**.

4. If

$$a_n = 3 + \frac{(\ln n)^6}{\sqrt{n}},$$

then the sequence a_n:

(a) Converges to 3.
(b) Diverges to infinity.
(c) Diverges, but not to infinity.
(d) Converges to 6.
(e) Converges to 4.

We evaluate

$$\lim_{n\to\infty} \left(3 + \frac{(\ln n)^6}{\sqrt{n}}\right)$$

Polynomial growth dominates logarithmic growth.

Therefore,

$$\lim_{n\to\infty} \frac{(\ln n)^6}{\sqrt{n}} = 0$$

$$\boxed{\lim_{n\to\infty} a_n = 3}$$

Answer: **(a)**.

5. Which of the following expressions represents the volume generated by rotating the region bounded by the curve

$$y = 1 + x^2 - 2x^4$$

and the x-axis, for $0 \leq x \leq 1$, about the y-axis?

(a) $2\pi \displaystyle\int_0^1 x\sqrt{1 + (2x - 8x^3)^2}\, dx$

(b) $2\pi \displaystyle\int_0^1 \sqrt{1 + (2x - 8x^3)^2}\, dx$

(c) $2\pi \displaystyle\int_0^1 (x + x^3 - 2x^5)\, dx$

(d) $2\pi \displaystyle\int_0^1 (1 + x^2 - 2x^4)\, dx$

(e) $2\pi \displaystyle\int_0^1 y\sqrt{1 + (2x - 8x^3)^2}\, dx$

Using the shell method,

$$V = 2\pi \int_a^b (\text{radius})(\text{height})\, dx$$

The radius is x and the height is

$$y = 1 + x^2 - 2x^4$$

Therefore,

$$V = 2\pi \int_0^1 x(1 + x^2 - 2x^4)\, dx$$

$$\boxed{V = 2\pi \int_0^1 (x + x^3 - 2x^5)\, dx}$$

Answer: **(c)**.

6. Find the sum of the geometric series:

$$12 + 12\left(\frac{7}{19}\right) + 12\left(\frac{7}{19}\right)^2 + 12\left(\frac{7}{19}\right)^3 + \cdots$$

(a) 20
(b) 8
(c) 19
(d) 14
(e) $\frac{7}{19}$

For a geometric series,

$$S = \frac{a}{1-r}, \qquad |r| < 1$$

$$a = 12, \qquad r = \frac{7}{19}$$

Therefore,

$$S = \frac{12}{1 - \frac{7}{19}}$$

$$\boxed{S = 19}$$

Answer: **(c)**.

7. Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{\cos^n(x)}{3^n}$$

(a) $\dfrac{3}{3-\cos(x)}$

(b) $\dfrac{1}{3-\cos(x)}$

(c) $\dfrac{2}{3-\cos(x)}$

(d) $\dfrac{3}{3-\sin(x)}$

(e) $\dfrac{1}{3-\sin(x)}$

Rewrite the series as

$$\sum_{n=0}^{\infty} \left(\frac{\cos(x)}{3}\right)^n$$

This is a geometric series with

$$a = 1, \qquad r = \frac{\cos(x)}{3}$$

Since $|r| \leq 1/3 < 1$, the series converges.

$$S = \frac{1}{1 - \frac{\cos(x)}{3}}$$

$$\boxed{S = \frac{3}{3-\cos(x)}}$$

Answer: **(a)**.

8. Determine the value of the series, if it converges:

$$\sum_{n=1}^{\infty} \frac{5^{n+2}}{6^n}$$

(a) 64

(b) The series diverges.

(c) 2

(d) 125

(e) 25

Rewrite the series:

$$\sum_{n=1}^{\infty} 25 \left(\frac{5}{6}\right)^n$$

$$a = \frac{125}{6}$$

$$r = \frac{5}{6}$$

Therefore,

$$S = \frac{\frac{125}{6}}{1 - \frac{5}{6}}$$

$$\boxed{S = 125}$$

Answer: **(d)**.

9. Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} e^{3/n}$$

(a) Converges to 0.

(b) Converges to 1.

(c) Diverges by the test for divergence.

(d) Converges to e^3.

(e) None of the above.

Using the test for divergence, evaluate the limit of the terms.

$$\lim_{n\to\infty} e^{3/n} = e^0 = 1$$

Since the limit is not zero, the series diverges.

$$\boxed{\text{Diverges}}$$

Answer: **(c)**.

10. Consider the region bounded above by both curves

$$y = \sqrt{4 - x^2}, \qquad y = 1$$

and below by the x-axis, for $-2 \leq x \leq 2$.

Find the volume of the solid generated by revolving this region about the x-axis.

(a) $\dfrac{32\pi}{3} - 4\pi\sqrt{3}$

(b) $\dfrac{32\pi}{3} - 2\pi\sqrt{3}$

(c) $\dfrac{16\pi}{3} - 4\pi\sqrt{3}$

(d) $4\pi\sqrt{3}$

(e) None of the above.

First, find the intersection points:

$$\sqrt{4 - x^2} = 1$$

$$4 - x^2 = 1$$

$$x = \pm\sqrt{3}$$

For $-\sqrt{3} \leq x \leq \sqrt{3}$, the radius is 1.

For $\sqrt{3} \leq |x| \leq 2$, the radius is $\sqrt{4 - x^2}$.

By symmetry, the disk method gives

$$V = 2\pi \left[\int_0^{\sqrt{3}} 1^2 \, dx + \int_{\sqrt{3}}^{2} (4 - x^2) \, dx \right]$$

$$V = 2\pi \left[\sqrt{3} + \left(4x - \frac{x^3}{3} \right) \Big|_{\sqrt{3}}^{2} \right]$$

$$V = 2\pi \left[\sqrt{3} + \frac{16}{3} - 3\sqrt{3} \right]$$

$$\boxed{V = \frac{32\pi}{3} - 4\pi\sqrt{3}}$$

Answer: **(a)**.

11. Set up the integral for the surface area generated by revolving

$$y = \sqrt{9 - x^2}, \qquad 0 \le x \le 3$$

about the y-axis.

(a) $2\pi \displaystyle\int_0^3 x\sqrt{1 + \frac{x^2}{9 - x^2}}\, dx$

(b) $2\pi \displaystyle\int_0^3 \sqrt{9 - x^2}\, dx$

(c) $\pi \displaystyle\int_0^3 (9 - x^2)\, dx$

(d) $2\pi \displaystyle\int_0^3 x\, dx$

(e) None of the above.

For rotation about the y-axis,

$$S = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y' = \frac{-x}{\sqrt{9 - x^2}}$$

$$1 + (y')^2 = 1 + \frac{x^2}{9 - x^2}$$

The surface area is

$$\boxed{S = 2\pi \int_0^3 x\sqrt{1 + \frac{x^2}{9 - x^2}}\, dx}$$

Answer: **(a)**.

12. Find the average value of the function

$$f(x) = x^2 + 2x$$

on the interval $[0, 3]$.

(a) 3
(b) 6
(c) 9
(d) 12
(e) None of the above.

The average value of a function is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Thus,

$$f_{\text{avg}} = \frac{1}{3}\int_0^3 (x^2 + 2x)\,dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3} + x^2\right]_0^3$$

$$= \frac{1}{3}(9 + 9)$$

$$\boxed{f_{\text{avg}} = 6}$$

Answer: **(b)**.

13. Determine whether the following sequence converges or diverges. If it converges, determine its limit.

$$a_n = \frac{4n^2 + 3n}{2n^2 + 5}$$

(a) Converges to 0.

(b) Converges to 2.

(c) Converges to 4.

(d) Diverges to infinity.

(e) None of the above.

Divide the numerator and denominator by n^2.

$$a_n = \frac{4 + \frac{3}{n}}{2 + \frac{5}{n^2}}$$

As $n \to \infty$,

$$\frac{3}{n} \to 0, \qquad \frac{5}{n^2} \to 0$$

Therefore,

$$\lim_{n\to\infty} a_n = \frac{4}{2}$$

$$\boxed{\lim_{n\to\infty} a_n = 2}$$

Answer: **(b)**.

14. Use the Integral Test to determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$$

(a) Converges because the corresponding improper integral converges.

(b) Diverges because the corresponding improper integral diverges.

(c) Diverges because the terms approach zero.

(d) The Integral Test cannot be applied.

(e) None of the above.

Consider

$$f(x) = \frac{1}{(x+2)^2}$$

The function is positive, continuous, and decreasing for $x \geq 1$.

Therefore, the Integral Test applies.

Evaluate

$$\int_1^{\infty} \frac{1}{(x+2)^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b (x+2)^{-2}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{x+2}\right]_1^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b+2} + \frac{1}{3}\right)$$

$$= \frac{1}{3}$$

Since the improper integral converges, the series also converges by the Integral Test.

$$\boxed{\text{Converges}}$$

Answer: **(a)**.