Math 220
Midterm 2 Review



General Differentiation Rules

Sum rule	$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
Difference rule	$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
Constant multiple rule	$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x)$
Constant rule	$\frac{d}{dx}k = 0$

$\frac{d}{dx}c = 0$	Constant Rule
$\frac{d}{dx}x^n = nx^{n-1}$	Power Rule
$\frac{d}{dx}\sin(x) = \cos(x)$	Trigonometric Rules
$\frac{d}{dx}\cos(x) = -\sin(x)$	
$\frac{d}{dx}b^x = b^x \ln(b)$	Exponential Rule
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	Logarithmic Rule

Product & Quotient Rules

Product Rule



$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient Rule



$$f'(x) = \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)\,v(x) - u(x)v'(x)}{[v(x)]^2}$$

Trig Derivative Rules

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Trig Derivative Rules

DERIVATIVES OF INVERSE TRIG FUNCTIONS

$y = \arcsin x \quad y = \sin^{-1} x$

$\sin y = x$

$\frac{d}{dx}\sin y = \frac{d}{dx}x$

$\cos y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}}$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

$y = \arccos x \quad y = \cos^{-1} x$

$\cos y = x$

$\frac{d}{dx}\cos y = \frac{d}{dx}x$

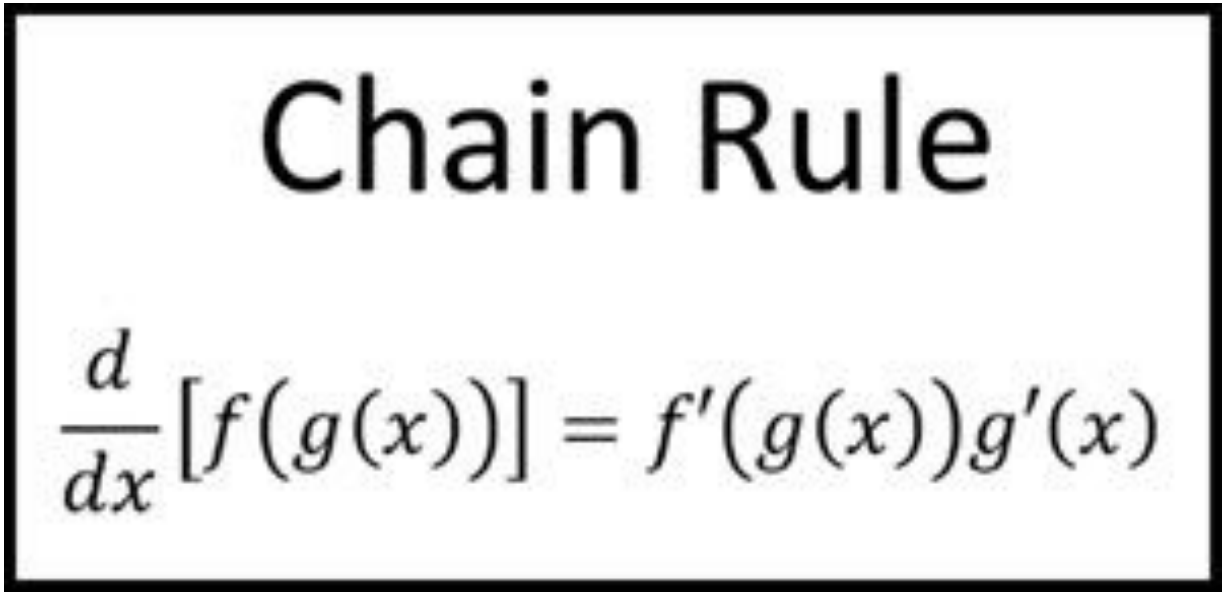
$-\sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}}$

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1 - x^2}}$$

If you don't remember the derivative of an inverse trig function, you can derive it yourself!

Chain Rule



Implicit vs. Explicit Differentiation

- An implicit function is a function, written in terms of both dependent and independent variables
- An explicit function is a function which is represented in terms of an independent variable.

$$xy = 1$$

$$y = \frac{1}{x} = x^{-1}$$

Implicit Differentiation

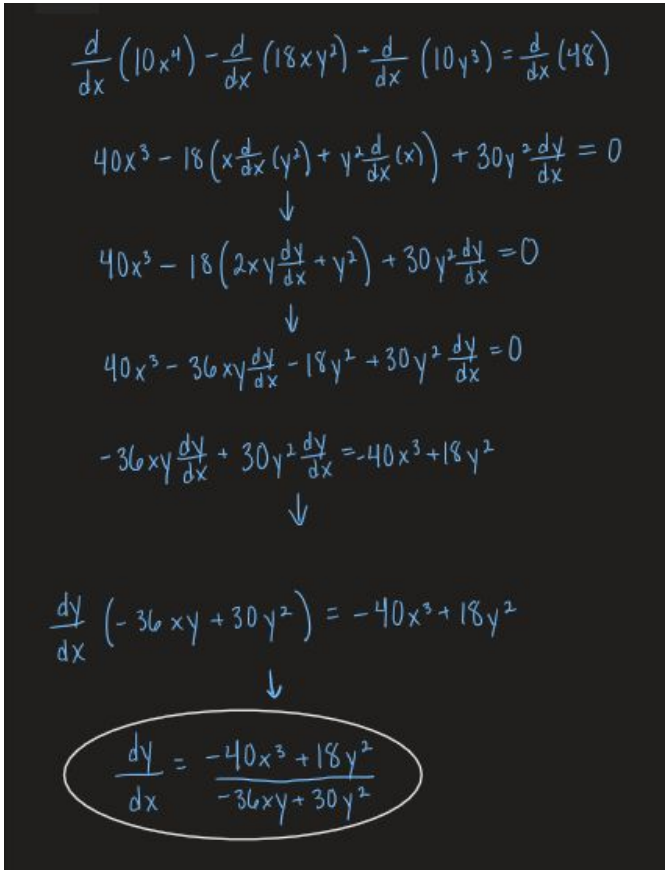
The technique of implicit differentiation allows you to find the derivative of y with respect to x without having to solve the given equation for y.

Solve for dy/dx in the example!

$$10x^4 - 18xy^2 + 10y^3 = 48$$

Implicit Differentiation Example Walkthrough

$$10x^4 - 18xy^2 + 10y^3 = 48$$



Related Rates

Steps to solving related rates problems:

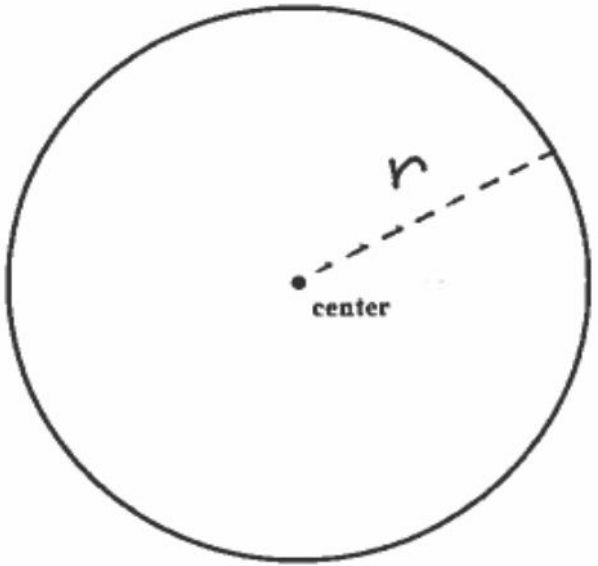
1. Draw and label a picture (with variables if possible)
2. Write an equation relating the variable whose change you want with the variables whose change you know
3. Differentiate both sides of the equation
4. Substitute values into equation

Related Rates Example

The radius of a circular oil slick on the surface of a pond is increasing at the rate of $10\ meters/min$. At what rate is the circle's

a.) circumference changing
b.) area changing

when the radius of the oil slick is $20\ m$. ?



Related Rates Example Walkthrough

a.) The circumference of a circle is

$$C = 2\pi r$$

GIVEN: $\frac{dr}{dt} = 10\ meters/min.$

FIND: $\frac{dC}{dt}$ when $r = 20\ m.$

Now differentiate the circumference equation with respect to time t getting

$$D\{C\} = D\{2\pi r\} \quad \longrightarrow$$

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt} \quad \longrightarrow$$

$\left(\text{Now let } \frac{dr}{dt} = 10.\right)$

$$\frac{dC}{dt} = 2\pi(10) \quad \longrightarrow$$

$$\frac{dC}{dt} = 20\pi\ \ m/min. \approx 62.8\ \ meters/min.$$

b.) The area of a circle is

$$A = \pi r^2$$

GIVEN: $\frac{dr}{dt} = 10\ meters/min.$

FIND: $\frac{dA}{dt}$ when $r = 20\ m.$

Now differentiate the area equation with respect to time t getting

$$D\{A\} = D\{\pi r^2\} \quad \longrightarrow$$

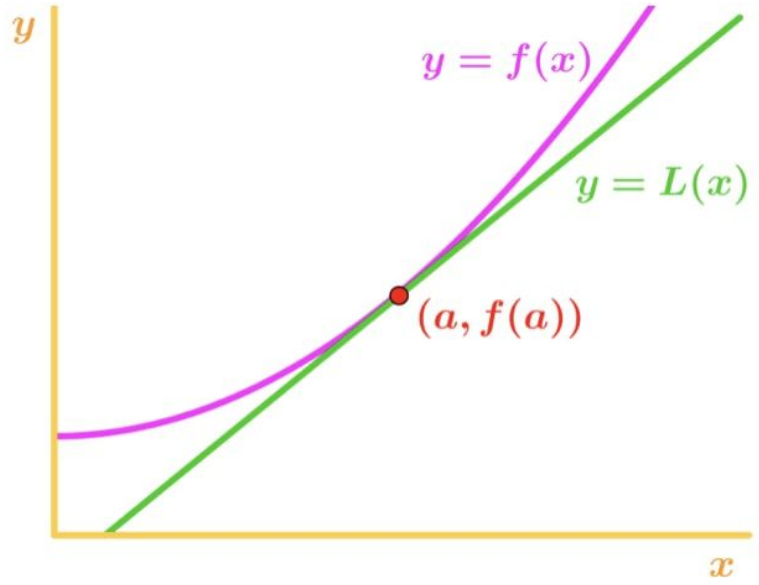
$$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt} \quad \longrightarrow$$

$\left(\text{Now let } \frac{dr}{dt} = 10 \text{ and } r = 20.\right)$

$$\frac{dA}{dt} = 2\pi(20)(10) = 400\pi\ \ meters^2/min. \approx 1256.6\ \ meters^2/min.$$

Linear Approximation/Linearization

$$L(x) = f(a) + f'(a)(x - a)$$



Linear Approximation Example

Find the linear approximation L(x) of $\sqrt[3]{x}$ at a = 1000.

Use this to approximate at x = 1012.

Linear Approximation Example Walkthrough

Step 1: Use the linearization formula

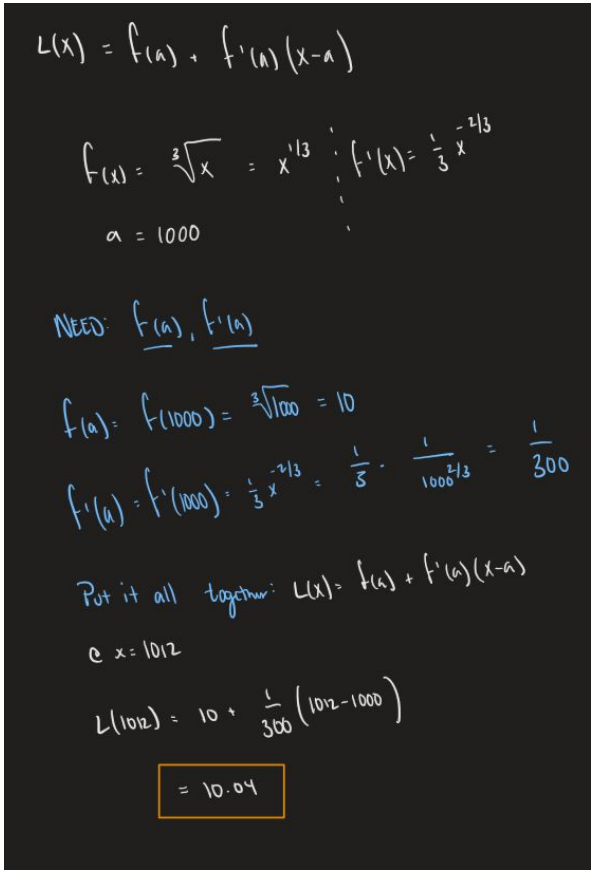
Step 2: Find f(a)

Step 3: Find the derivative

Step 4: Evaluate the **derivative** at a = 1000

Step 5: Write the linear approximation

Step 6: Use it to approximate $\sqrt[3]{1012}$



Exponential Growth and Decay

Discrete Form:

$$y = a(b)^{nt}$$

Language:

- Per month/year
- Every quarter

Continuous Form:

$$y = ae^{kt}$$

$$\frac{dy}{dx} = ky$$

Language:

- Constantly changing
- Continuous process
- Proportional to size

Where **y(t)** = value at time "t"
a = value at the start
k = rate of growth (when >0) or decay (when <0)
t = time

Exponential Growth/Decay Example

21. A population of bacteria initially has 90,000 present and in 2 weeks there will be 200,000 bacteria present.

(a) Determine the exponential growth equation for this population.

(b) How long will it take for the population to grow from its initial population of 90,000 to a population of 150,000?

Exponential Growth/Decay Example Walkthrough

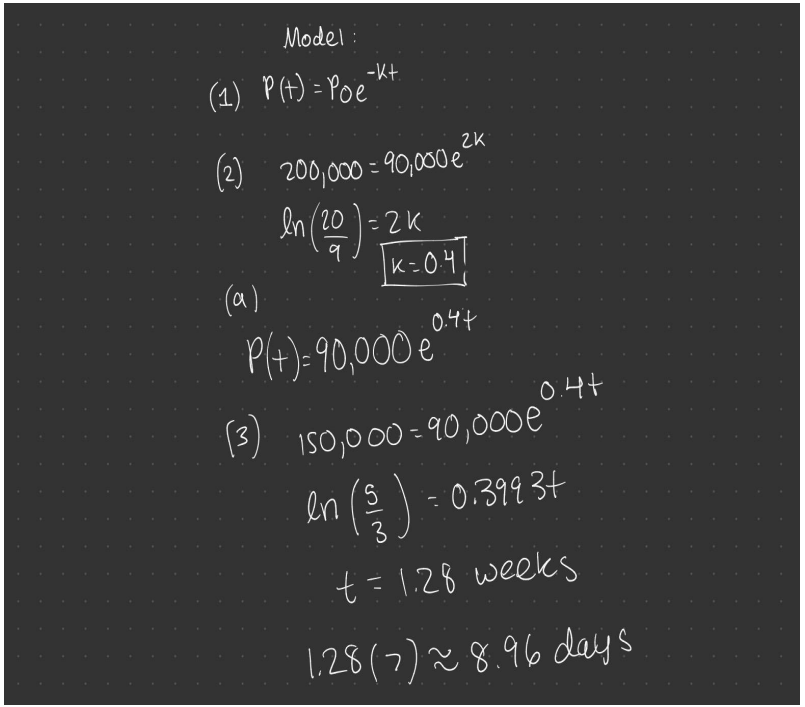
Step 1: Set up the exponential growth model

Step 2: Use the information after 2 weeks

Step 3: Solve for k

Step 4: Substitute back into the model

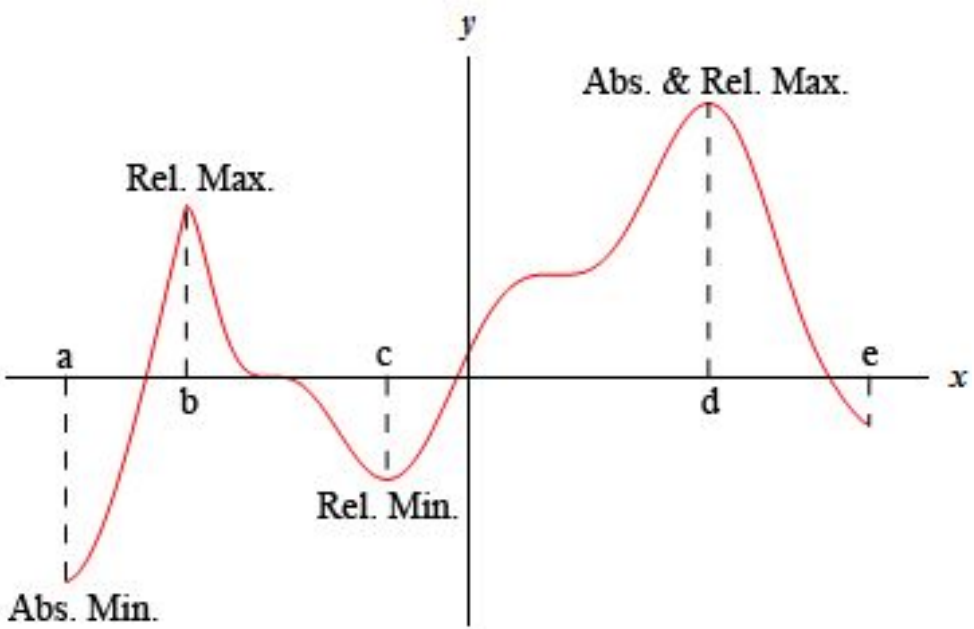
Step 5: Solve for t



Maximums and Minimums

Local maximum and minimum can only occur at critical values

Critical numbers are x values that will make the first derivative of your function equal to zero



Solving Max/Min Problems

Steps to solving problems:

1. Evaluate if boundary restrictions are included
2. Take derivative of function
3. Identify values that will make your derivative function equal to 0 (critical values)
4. Plug values back into original function
5. Evaluate magnitude of values compared to values for the boundary restrictions

Mean Value Theorem

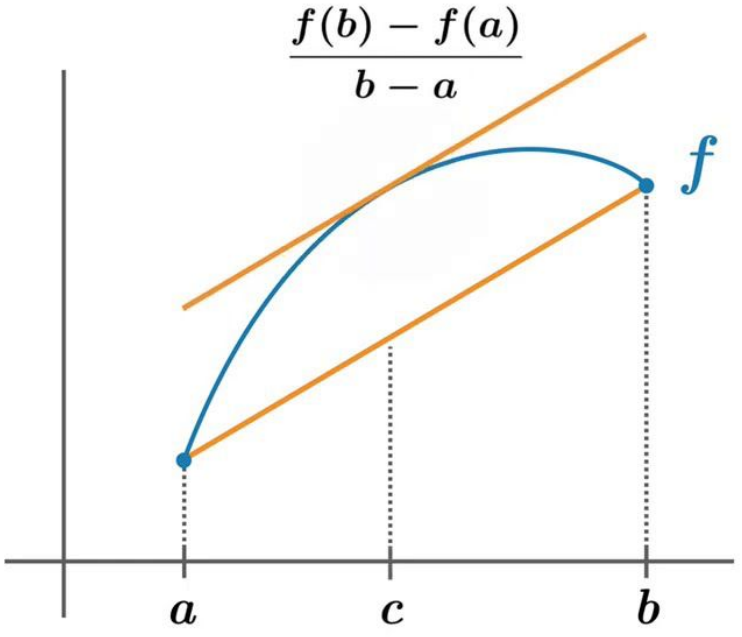
The Mean Value Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some argument c such that

$$a < c < b$$

and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



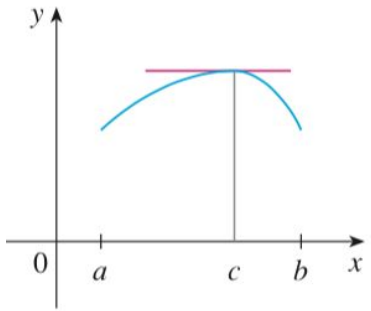
Rolle's Theorem

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

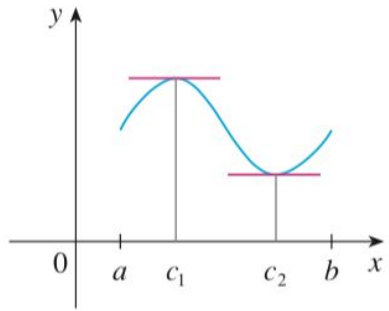
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

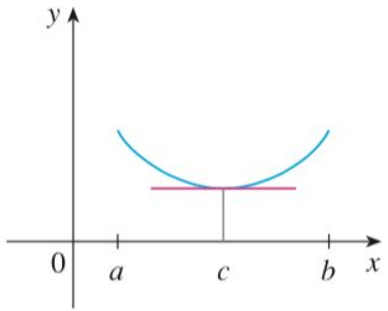
Before giving the proof let's take a look at the graphs of some typical functions that satisfy the three hypotheses. Figure 1 shows the graphs of four such functions. In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$. Thus Rolle's Theorem is plausible.



(b)



(c)



(d)

1st and 2nd Derivative Tests & Sketching

First derivative:

y' is positive ➡ Curve is rising.

y' is negative ➡ Curve is falling.

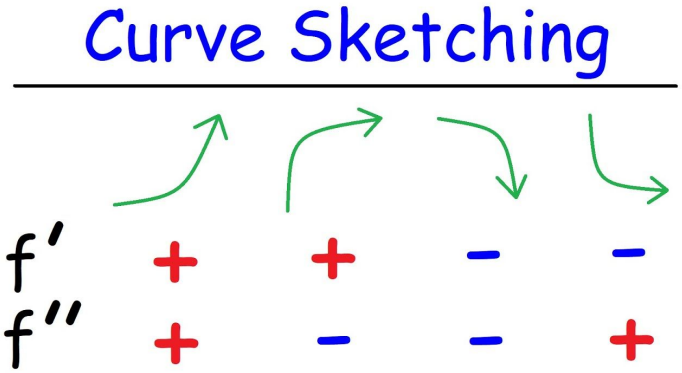
y' is zero ➡ Possible local maximum or minimum.

Second derivative:

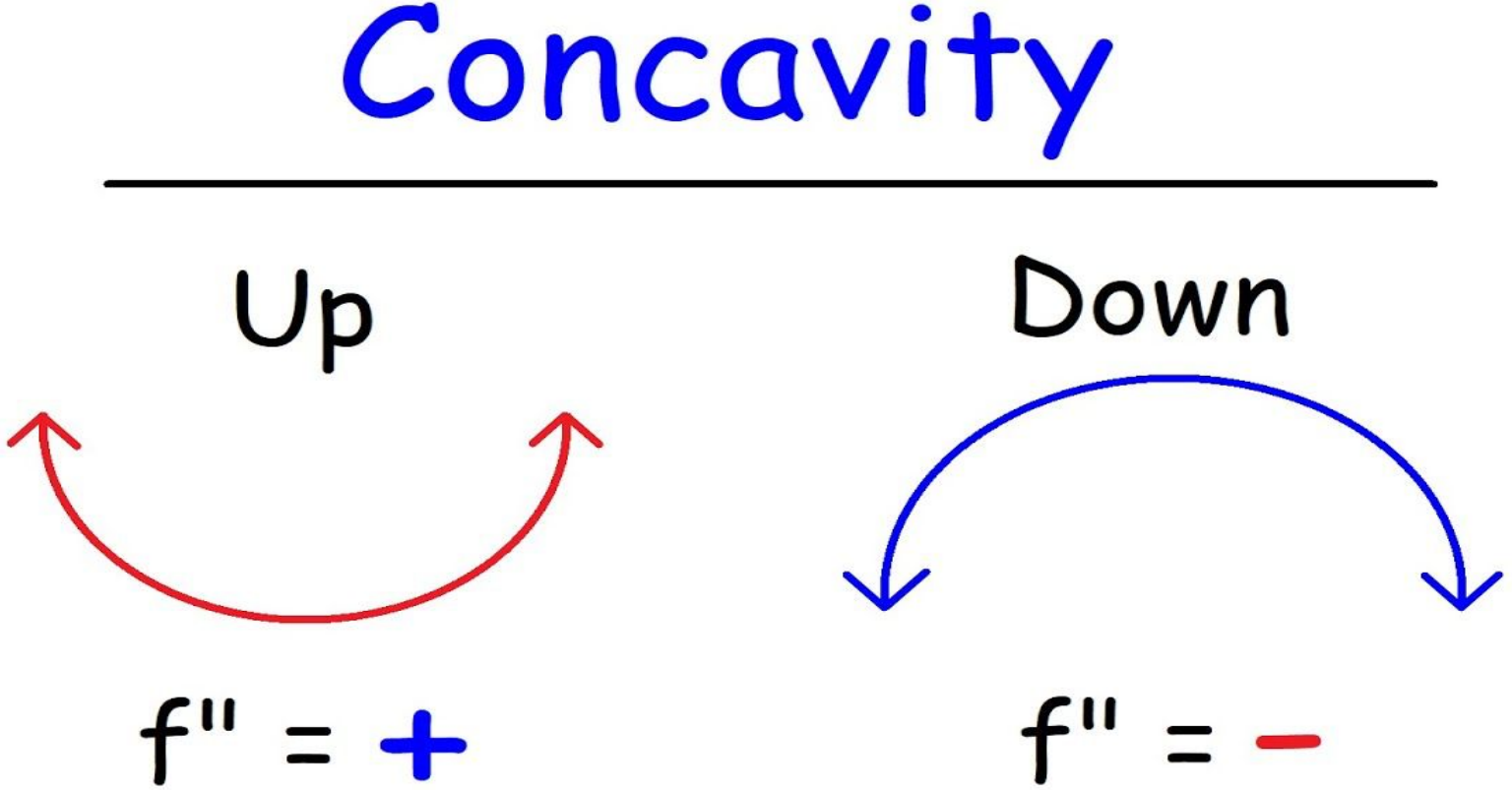
y'' is positive ➡ Curve is concave up. 

y'' is negative ➡ Curve is concave down. 

y'' is zero ➡ Possible inflection point (where concavity changes). 



Concavity



l'Hopital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval containing a, except possibly at a.

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if this limit exists.

A similar rule also works for one sided limits and limits as x goes to infinity.

Indeterminate Forms

Determinate-Indeterminate Forms Table	
Indeterminate Forms	Determinate Forms
$0/0$	$\infty + \infty = \infty$
$\pm\infty/\pm\infty$	$-\infty - \infty = -\infty$
$\infty - \infty$	$0^{\infty} = 0$
$0(\infty)$	$0^{-\infty} = \infty$
0^0	$(\infty) \cdot (\infty) = \infty$
1^{∞}	
∞^0	
Use L'Hôpital's Rule	Do *Not* Use L'Hôpital's Rule

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