



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 231E – Engineering Calculus

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Tutors: Oct. 11, 3-4:50pm Aadya and Amishi

Can't make it to a session? Here's our schedule by course:

<https://care.grainger.illinois.edu/tutoring/schedule-by-subject>

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: <https://queue.illinois.edu/q/queue/979>
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Evaluate

$$\int_0^\pi \sqrt{1 - \sin^2 x} \, dx.$$

- i. 0
- ii. 1
- iii. 2
- iv. π
- v. None of the above

$$\begin{aligned} \sqrt{1 - \sin^2 x} &= \sqrt{\cos^2 x} = |\cos x| \\ \int_0^\pi |\cos x| \, dx &= \int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^\pi \cos x \, dx \\ &= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^\pi = 1 - (-1) = \boxed{2} \end{aligned}$$

Answer: (c).

2. Evaluate

$$\int_0^{\pi/2} (\sin^6 x + \cos^6 x) \, dx.$$

- i. $\frac{5\pi}{32}$
- ii. $\frac{3\pi}{8}$
- iii. $\frac{\pi}{2}$
- iv. $\frac{5\pi}{16}$
- v. None of the above

$$\begin{aligned} \sin^6 x + \cos^6 x &= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x) \\ &= 1 - \frac{3}{4} \sin^2(2x) = \frac{5}{8} + \frac{3}{8} \cos(4x) \\ \int_0^{\pi/2} \left(\frac{5}{8} + \frac{3}{8} \cos(4x) \right) \, dx &= \left[\frac{5x}{8} + \frac{3 \sin(4x)}{32} \right]_0^{\pi/2} = \boxed{\frac{5\pi}{16}} \end{aligned}$$

Answer: (d).

3. Evaluate

$$\int_0^{\pi/2} \frac{\sin^3 x \cos^5 x}{(1 + \sin^2 x)^2} dx.$$

- i. $4 \ln 2 - \frac{11}{4}$
- ii. $2 \ln 2 - \frac{5}{4}$
- iii. $4 \ln 2 - \frac{5}{2}$
- iv. $\frac{11}{4} - 4 \ln 2$
- v. None of the above

$$u = \sin^2 x, \quad du = 2 \sin x \cos x \, dx, \quad \cos^4 x = (1 - u)^2$$

$$x = 0 \Rightarrow u = 0, \quad x = \frac{\pi}{2} \Rightarrow u = 1$$

$$I = \frac{1}{2} \int_0^1 \frac{u(1-u)^2}{(1+u)^2} du$$

$$\frac{u(1-u)^2}{(1+u)^2} = u - 4 + \frac{8}{u+1} - \frac{4}{(u+1)^2}$$

$$I = \frac{1}{2} \left[\frac{u^2}{2} - 4u + 8 \ln(u+1) + \frac{4}{u+1} \right]_0^1$$

$$= \frac{1}{2} \left(\frac{1}{2} - 4 + 8 \ln 2 + 2 - 4 \right) = \boxed{4 \ln 2 - \frac{11}{4}}$$

Answer: (a).

4. Test the integral for convergence. If it converges, find its value.

$$\int_0^{\pi/2} \frac{1 - \cos(2x)}{1 + \cos(2x)} dx.$$

- i. Converges, value is 0
- ii. Converges, value is $1 - \frac{\pi}{2}$
- iii. Diverges
- iv. Converges, value is $\frac{\pi}{2}$
- v. None of the above

$$\frac{1 - \cos(2x)}{1 + \cos(2x)} = \frac{2 \sin^2 x}{2 \cos^2 x} = \tan^2 x$$

$$\lim_{b \rightarrow (\pi/2)^-} \int_0^b (\sec^2 x - 1) dx = \lim_{b \rightarrow (\pi/2)^-} [\tan x - x]_0^b = +\infty$$

Diverges

Answer: (c).

5. Evaluate

$$\int \frac{dx}{(x^2 + 4x + 8)^{3/2}}.$$

- i. $\frac{x+2}{2\sqrt{x^2+4x+8}} + C$
- ii. $\frac{x}{4\sqrt{x^2+4x+8}} + C$
- iii. $\frac{x+2}{4\sqrt{x^2+4x+8}} + C$
- iv. $-\frac{1}{2\sqrt{x^2+4x+8}} + C$
- v. None of the above

$$x^2 + 4x + 8 = (x+2)^2 + 4$$

$$x+2 = 2 \tan \theta, \quad dx = 2 \sec^2 \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\int \frac{2 \sec^2 \theta}{8 \sec^3 \theta} d\theta = \frac{1}{4} \int \cos \theta d\theta = \frac{1}{4} \sin \theta + C$$

$$\sin \theta = \frac{x+2}{\sqrt{x^2+4x+8}}$$

$$\boxed{\frac{x+2}{4\sqrt{x^2+4x+8}} + C}$$

Answer: (c).

6. Evaluate

$$\int_0^3 \frac{x^2}{\sqrt{9-x^2}} dx.$$

- i. $\frac{3\pi}{4}$
- ii. $\frac{9\pi}{2}$
- iii. Diverges
- iv. $\frac{9\pi}{4}$
- v. None of the above

$$x = 3 \sin \theta, \quad dx = 3 \cos \theta \, d\theta, \quad 0 \leq \theta < \frac{\pi}{2}$$

$$\sqrt{9-x^2} = 3 \cos \theta$$

$$I = 9 \lim_{b \rightarrow (\pi/2)^-} \int_0^b \sin^2 \theta \, d\theta$$

$$= \frac{9}{2} \lim_{b \rightarrow (\pi/2)^-} \int_0^b (1 - \cos 2\theta) \, d\theta$$

$$= \frac{9}{2} \lim_{b \rightarrow (\pi/2)^-} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^b = \boxed{\frac{9\pi}{4}}$$

Answer: (d).

7. For $x < -4$, use $x = 4 \sec \theta$ with $\pi/2 < \theta < \pi$. Which expression equals $\sqrt{x^2 - 16}$?

- i. $4 \tan \theta$
- ii. $-4 \tan \theta$
- iii. $4 \sec \theta$
- iv. $-4 \sec \theta$
- v. None of the above

$$\sqrt{x^2 - 16} = \sqrt{16 \sec^2 \theta - 16} = 4\sqrt{\tan^2 \theta} = 4|\tan \theta|$$

$$\frac{\pi}{2} < \theta < \pi \quad \Rightarrow \quad \tan \theta < 0$$

$$\boxed{-4 \tan \theta}$$

Answer: (b).

8. Which is the complete general partial-fraction form?

$$\frac{x^4 + 1}{x(x-1)^2(x^2 + 1)}$$

- i. $\frac{A}{x} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 1}$
- ii. $\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{x^2 + 1}$
- iii. $\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx + E}{x^2 + 1}$
- iv. $A + \frac{B}{x} + \frac{C}{x-1} + \frac{Dx + E}{x^2 + 1}$
- v. None of the above

The numerator has degree 4 and the denominator has degree 5. No polynomial division is needed. The repeated factor $(x-1)^2$ requires both powers. The irreducible quadratic requires a general linear numerator.

$$\boxed{\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx + E}{x^2 + 1}}$$

Answer: (c).

9. Evaluate

$$\int \frac{x^2 + 1}{x(x-1)^2} dx.$$

- i. $\ln|x| + \frac{2}{x-1} + C$
- ii. $\ln|x| - \frac{2}{x-1} + C$
- iii. $\ln|x| + 2\ln|x-1| - \frac{2}{x-1} + C$
- iv. $x + \ln|x| - \frac{2}{x-1} + C$
- v. None of the above

$$\frac{x^2 + 1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{D}{(x-1)^2}$$

$$x^2 + 1 = A(x-1)^2 + Bx(x-1) + Dx$$

$$x = 0 : A = 1, \quad x = 1 : D = 2, \quad A + B = 1 : B = 0$$

$$\int \left(\frac{1}{x} + \frac{2}{(x-1)^2} \right) dx = \boxed{\ln|x| - \frac{2}{x-1} + C}$$

Answer: (b).

10. Evaluate

$$\int \frac{x^3 + 1}{(x-1)^2(x^2+1)} dx.$$

- i. $\frac{1}{2} \ln|x-1| + \frac{1}{x-1} + \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \arctan x + C$
- ii. $\frac{1}{2} \ln|x-1| - \frac{1}{x-1} + \frac{1}{2} \ln(x^2+1) + \frac{1}{2} \arctan x + C$
- iii. $\frac{1}{2} \ln|x-1| - \frac{1}{x-1} + \frac{1}{4} \ln(x^2+1) - \frac{1}{2} \arctan x + C$
- iv. $\frac{1}{2} \ln|x-1| - \frac{1}{x-1} + \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \arctan x + C$
- v. None of the above

$$\begin{aligned} \frac{x^3 + 1}{(x-1)^2(x^2+1)} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Dx+E}{x^2+1} \\ x^3 + 1 &= A(x-1)(x^2+1) + B(x^2+1) + (Dx+E)(x-1)^2 \\ A+D &= 1, \quad -A+B-2D+E=0 \\ A+D-2E &= 0, \quad -A+B+E=1 \\ A &= \frac{1}{2}, \quad B=1, \quad D=\frac{1}{2}, \quad E=\frac{1}{2} \\ I &= \int \left(\frac{1}{2(x-1)} + \frac{1}{(x-1)^2} + \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)} \right) dx \\ &= \boxed{\frac{1}{2} \ln|x-1| - \frac{1}{x-1} + \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \arctan x + C} \end{aligned}$$

Answer: (d).

11. Test the integral for convergence. If it converges, find its value.

$$\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)^2}.$$

- i. Converges, value is $\frac{\pi-2}{8}$
- ii. Converges, value is $\frac{\pi+1}{4}$
- iii. Diverges
- iv. Converges, value is $\frac{\pi-1}{4}$
- v. None of the above

$$\begin{aligned} \frac{1}{(x+1)(x^2+1)^2} &= \frac{A}{x+1} + \frac{Bx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} \\ 1 &= A(x^2+1)^2 + (Bx+D)(x+1)(x^2+1) + (Ex+F)(x+1) \\ A &= \frac{1}{4}, \quad B = -\frac{1}{4}, \quad D = \frac{1}{4}, \quad E = -\frac{1}{2}, \quad F = \frac{1}{2} \end{aligned}$$

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{1}{4(x+1)} - \frac{x}{4(x^2+1)} + \frac{1}{4(x^2+1)} - \frac{x}{2(x^2+1)^2} + \frac{1}{2(x^2+1)^2}$$

For the last term:

$$\begin{aligned} x &= \tan \theta, \quad \int \frac{dx}{(x^2+1)^2} = \int \cos^2 \theta \, d\theta \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{1}{2} \arctan x + \frac{x}{2(x^2+1)} + C \end{aligned}$$

Combining the terms:

$$I = \lim_{b \rightarrow \infty} \left[\frac{1}{4} \ln(x+1) - \frac{1}{8} \ln(x^2+1) + \frac{1}{2} \arctan x + \frac{x+1}{4(x^2+1)} \right]_0^b$$

$$\frac{1}{4} \ln(b+1) - \frac{1}{8} \ln(b^2+1) = \frac{1}{8} \ln \left(\frac{(b+1)^2}{b^2+1} \right) \rightarrow 0$$

$$I = 0 + \frac{\pi}{4} + 0 - \frac{1}{4} = \boxed{\frac{\pi-1}{4}}$$

Answer: (d).

12. For which real values of p and q does the integral converge?

$$\int_0^\infty \frac{x^p}{|x-1|^q(1+x^2)} dx.$$

- i. $p > -1, \quad q < 1, \quad p < q + 1$
- ii. $p > -1, \quad q < 1$
- iii. $p \geq -1, \quad q \leq 1, \quad p \leq q + 1$
- iv. $p > -1, \quad q > 1, \quad p < q + 1$
- v. None of the above

Let

$$f(x) = \frac{x^p}{|x-1|^q(1+x^2)}.$$

At $x = 0$:

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x^p} = 1$$

$$\int_0^{1/2} x^p dx \text{ converges} \iff p > -1.$$

At $x = 1$:

$$\lim_{x \rightarrow 1} \frac{f(x)}{|x-1|^{-q}} = \frac{1}{2}$$

$$\int_0^{1/2} t^{-q} dt \text{ converges} \iff q < 1.$$

At infinity:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^{p-q-2}} = 1$$

$$\int_2^\infty x^{p-q-2} dx \text{ converges} \iff p - q - 2 < -1.$$

$$\boxed{p > -1, \quad q < 1, \quad p < q + 1}$$

Equality at any boundary gives a divergent integral of the form $\int dt/t$.

Answer: (a).

13. For which real value of a does the integral converge, and what is its value?

$$\int_0^\infty \left(\frac{x^2}{(x^2 + 4)^{3/2}} - \frac{a}{x + 2} \right) dx.$$

- i. $a = 1$, value is $\ln 2$
- ii. $a = 1$, value is $\ln 2 - 1$
- iii. $a = \frac{1}{2}$, value is $\frac{1}{2} \ln 2 - 1$
- iv. Diverges for every real a
- v. None of the above

$$\begin{aligned} x &= 2 \tan \theta, & dx &= 2 \sec^2 \theta d\theta, & 0 &\leq \theta < \frac{\pi}{2} \\ \int \frac{x^2}{(x^2 + 4)^{3/2}} dx &= \int \frac{\tan^2 \theta}{\sec \theta} d\theta = \int (\sec \theta - \cos \theta) d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C \\ &= \ln \left(\frac{x + \sqrt{x^2 + 4}}{2} \right) - \frac{x}{\sqrt{x^2 + 4}} + C \\ I &= \lim_{b \rightarrow \infty} \left[\ln \left(\frac{b + \sqrt{b^2 + 4}}{2} \right) - \frac{b}{\sqrt{b^2 + 4}} - a \ln(b + 2) + a \ln 2 \right] \\ &= \lim_{b \rightarrow \infty} \left[(1 - a) \ln b + \ln \left(\frac{1 + \sqrt{1 + 4/b^2}}{2} \right) - \frac{1}{\sqrt{1 + 4/b^2}} - a \ln(1 + 2/b) + a \ln 2 \right] \end{aligned}$$

The limit is finite only when $a = 1$.

$$I = 0 + 0 - 1 - 0 + \ln 2 = \boxed{\ln 2 - 1}$$

Answer: (b).

14. Which statement is correct?

$$I = \int_2^{\infty} \frac{dx}{x\sqrt{x^2-4}}, \quad J = \int_2^{\infty} \frac{dx}{x^2-4}.$$

- i. Both converge.
- ii. I converges and J diverges.
- iii. I diverges and J converges.
- iv. Both diverge.
- v. None of the above

For I :

$$x = 2 \sec \theta, \quad dx = 2 \sec \theta \tan \theta d\theta, \quad 0 < \theta < \frac{\pi}{2}$$

$$\frac{dx}{x\sqrt{x^2-4}} = \frac{1}{2} d\theta$$

$$\begin{aligned} I &= \frac{1}{2} \lim_{\alpha \rightarrow 0^+} \int_{\alpha}^{\pi/4} d\theta + \frac{1}{2} \lim_{\beta \rightarrow (\pi/2)^-} \int_{\pi/4}^{\beta} d\theta \\ &= \frac{\pi}{8} + \frac{\pi}{8} = \frac{\pi}{4} \end{aligned}$$

For J :

$$\begin{aligned} \frac{1}{x^2-4} &= \frac{1}{4} \left(\frac{1}{x-2} - \frac{1}{x+2} \right) \\ \lim_{a \rightarrow 2^+} \int_a^3 \frac{dx}{x^2-4} &= \frac{1}{4} \lim_{a \rightarrow 2^+} \left[\ln \left| \frac{x-2}{x+2} \right| \right]_a^3 \\ &= \frac{1}{4} \lim_{a \rightarrow 2^+} \left(\ln \frac{1}{5} - \ln \frac{a-2}{a+2} \right) = +\infty \end{aligned}$$

I converges and J diverges

Answer: (b).

15. Evaluate using trigonometric substitution and partial fractions.

$$\int_0^{\sqrt{5}} \frac{x\sqrt{x^2+4}}{(x^2+1)^2(x^2+3)} dx.$$

- i. $\frac{3}{8} + \frac{\ln(2+\sqrt{3})}{4\sqrt{3}} + \frac{1}{8} \ln\left(\frac{3}{2}\right)$
- ii. $\frac{3}{8} - \frac{\ln(2+\sqrt{3})}{4\sqrt{3}} + \frac{1}{8} \ln\left(\frac{3}{2}\right)$
- iii. $\frac{3}{8} - \frac{\ln(2+\sqrt{3})}{4\sqrt{3}} - \frac{1}{8} \ln\left(\frac{3}{2}\right)$
- iv. $\frac{3}{4} - \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} + \frac{1}{4} \ln\left(\frac{3}{2}\right)$
- v. None of the above

$$x = 2 \tan \theta, \quad dx = 2 \sec^2 \theta d\theta, \quad \sqrt{x^2+4} = 2 \sec \theta$$

$$0 \leq \theta \leq \arctan\left(\frac{\sqrt{5}}{2}\right)$$

$$I = \int_0^{\arctan(\sqrt{5}/2)} \frac{8 \tan \theta \sec^3 \theta}{(4 \tan^2 \theta + 1)^2 (4 \tan^2 \theta + 3)} d\theta$$

$$4 \tan^2 \theta + 1 = 4 \sec^2 \theta - 3, \quad 4 \tan^2 \theta + 3 = 4 \sec^2 \theta - 1$$

$$u = 2 \sec \theta, \quad du = 2 \sec \theta \tan \theta d\theta$$

$$x = 0 \Rightarrow u = 2, \quad x = \sqrt{5} \Rightarrow u = 3$$

$$I = \int_2^3 \frac{u^2}{(u^2-3)^2(u^2-1)} du$$

First split the rational expression:

$$\frac{u^2}{(u^2-3)^2(u^2-1)} = \frac{A}{u^2-3} + \frac{B}{(u^2-3)^2} + \frac{D}{u^2-1}$$

$$u^2 = A(u^2-3)(u^2-1) + B(u^2-1) + D(u^2-3)^2$$

$$u^2 = 3 : B = \frac{3}{2}, \quad u^2 = 1 : D = \frac{1}{4}, \quad A + D = 0 : A = -\frac{1}{4}$$

$$\frac{u^2}{(u^2-3)^2(u^2-1)} = -\frac{1}{4(u^2-3)} + \frac{3}{2(u^2-3)^2} + \frac{1}{4(u^2-1)}$$

Let $r = \sqrt{3}$. Factoring the remaining denominators gives

$$\frac{u^2}{(u^2-3)^2(u^2-1)} = -\frac{1}{4r} \left(\frac{1}{u-r} - \frac{1}{u+r} \right) + \frac{1}{8} \left(\frac{1}{(u-r)^2} + \frac{1}{(u+r)^2} \right) + \frac{1}{8} \left(\frac{1}{u-1} - \frac{1}{u+1} \right)$$

$$I = \left[-\frac{1}{4r} \ln \left(\frac{u-r}{u+r} \right) - \frac{1}{8} \left(\frac{1}{u-r} + \frac{1}{u+r} \right) + \frac{1}{8} \ln \left(\frac{u-1}{u+1} \right) \right]_2^3$$

$$= \left[-\frac{1}{4r} \ln \left(\frac{u-r}{u+r} \right) - \frac{u}{4(u^2-3)} + \frac{1}{8} \ln \left(\frac{u-1}{u+1} \right) \right]_2^3$$

$$\frac{(3-r)(2+r)}{(3+r)(2-r)} = 2+r, \quad \frac{(3-1)/(3+1)}{(2-1)/(2+1)} = \frac{3}{2}$$

$$I = -\frac{\ln(2+r)}{4r} + \left(-\frac{1}{8} + \frac{1}{2} \right) + \frac{1}{8} \ln \left(\frac{3}{2} \right)$$

$$\boxed{\frac{3}{8} - \frac{\ln(2+\sqrt{3})}{4\sqrt{3}} + \frac{1}{8} \ln \left(\frac{3}{2} \right)}$$

Answer: (b).