

MATH 257 Exam 2 CARE Review

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In-Person Resources

CARE Drop-in tutoring:

7 days a week on the 4th floor
of Grainger Library!

Sunday - Thursday 12pm-10pm

Friday & Saturday 12-6pm

Course Office hours:

TAs: In-person: Mondays - Thursdays 5-7 pm

Focused on lectures and labs

CAs: On Zoom, schedule on Canvas.

Focused on Python

Instructors:

Chuang: Monday, 3-4pm, 233 CAB

Janda: Mondays 4-5pm, Harker Hall 304C

Baryshnikov: Tuesdays, 9:30-10:30 am

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	3pm-8pm	12pm-7pm	4pm-8pm	12pm-7pm	12pm-5pm 6pm-8pm	12pm- 4pm	2pm-6pm

Topic Summary

- LU Decomposition
 - Lower/Upper Triangular Matrix
 - LU for Linear Systems
 - Permutation Matrix
- Orthogonal Matrices
 - Inner Product
 - Orthogonality
- Subspaces
 - Column Space
 - Null Space
 - Linear Independence
 - Fundamental Subspaces
- Basis and Dimension
 - Fundamental Subspaces
 - Orthonormal bases
- Graph and Adjacency Matrices
- Coordinates
 - Coordinate Matrices
 - Orthogonal/normal Complements

Upper/Lower Triangular Matrices

Upper Triangular:

$$\begin{bmatrix} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \star \end{bmatrix}$$

Finding this is like
doing REF with only
row replacement

Lower Triangular:

$$\begin{bmatrix} \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \ddots & \vdots \\ \star & \star & \star & \star & \star \end{bmatrix}$$

Keep track of your row
operations to find L

LU Decomposition:

$$A = LU$$

- Not all matrices have LU decompositions
- LU decompositions are not unique (unlike inverses)

Finding the LU Decomposition

Determine the LU -decomposition of $\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix} \xrightarrow{\substack{1) \text{ Col 1 Row 2} \\ R_2 \rightarrow R_2 - 4R_1}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 4 & 4 & 8 \end{bmatrix} \xrightarrow{\substack{2) \text{ Col 1 Row 3} \\ R_3 \rightarrow R_3 - 4R_1}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & -4 & 0 \end{bmatrix} \xrightarrow{\substack{3) \text{ Col 3 Row 3} \\ R_3 \rightarrow R_3 - R_2}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$L := \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

$$U := \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

LU for Linear Systems

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_U$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_L \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}}_b$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_U \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}}_c$$

Use **LU decomposition** to solve a linear system if:

1. A is $n \times n$ matrix
2. $A = LU$
3. $b \in \mathbb{R}^n$

Step-by-step Algorithm

1. Find L and U
2. Solve for c using $Lc = b$
3. Solve for x using $Ux = c$

$$Ax = b$$

$$Lc = b \rightarrow Ux = c$$

$$Ax = (LU)x = L(Ux) = Lc = b$$

You Try: Solving with LU Decomposition

$$\text{Given : } A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\text{Solve } A\vec{x} = \vec{b} \text{ where } b = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Do not compute A

provide values of $\vec{x}$ and the intermediate vector $\vec{c}$

For your reference:

Step-by-step Algorithm

1. Find L and U
2. Solve for c using $Lc = b$
3. Solve for x using $Ux = c$

$$\begin{array}{c} A\mathbf{x} = \mathbf{b} \\ \swarrow \quad \searrow \\ L\mathbf{c} = \mathbf{b} \quad \rightarrow \quad U\mathbf{x} = \mathbf{c} \end{array}$$
$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = L\mathbf{c} = \mathbf{b}$$

Solution: Solving with LU Decomposition

Begin $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

I Substitute $U\vec{x} = \vec{c}$

II Solve $L\vec{c} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \xrightarrow{\text{fwd. sub.}} \vec{c} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

III Solve $U\vec{x} = \vec{c}$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \xrightarrow{\text{back sub.}} \vec{x} = \begin{bmatrix} -1 \\ -6 \\ -2 \end{bmatrix}$$

Permutation Matrices: for matrices that don't have an LU decomposition

Theorem 21. Let A be $n \times n$ matrix. Then there is a permutation matrix P such that PA has an LU-decomposition.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{\sim} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = PA$$
$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Step-by-step:

- Use the interchange operation done on A to an equivalent size identity matrix, this will be your P matrix
- Solve for the LU decomposition of PA

When we apply the P^{-1} to LU (on the right), we'll be able to get the original value of A

$$PA = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:U}$$

Inner Product, Norm, and Distance

$$\text{If } \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \text{ and } \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \text{ then } \mathbf{v} \cdot \mathbf{w} \text{ is}$$

$$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

The **inner product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

AKA the dot product

It is a scalar!

Definition. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

The **norm** (or **length**) of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The **distance** between $\mathbf{v}$ and $\mathbf{w}$ is

$$\text{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

The norm is also a scalar!

Properties of the Inner Product: similar to scalars

Theorem 22. *Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be any scalar. Then*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative!

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ Distributive!

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ Associative!

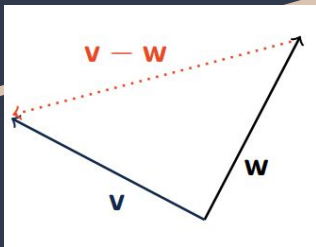
(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Orthogonality

(fancy word for perpendicular)

Vectors are orthogonal if their **dot product is zero**.

Why? The dot product of two non-zero vectors can only be zero if the angle between them is 90.



Orthonormality

A **unit vector** in $\mathbb{R}^n$ is vector of length 1.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

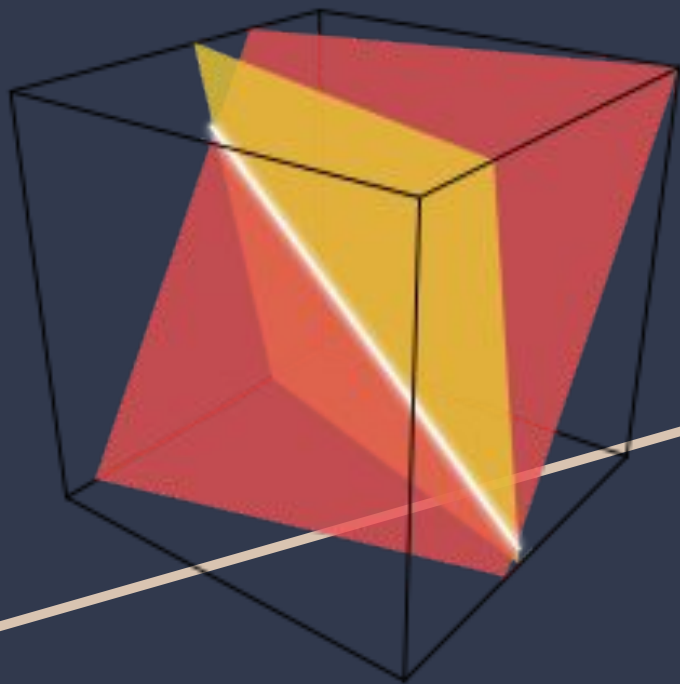
Orthonormal sets are all orthogonal to each other and unit vectors.

Ex.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Subspaces



W is a **subspace** of **V**, if:

- W contains the 0 vector
- Adding any 2 vectors in W together gives a vector also in W
- Multiplying any vector in W by any scalar gives a vector also in W

Theorem 24. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$.
Then $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ is a subspace of $\mathbb{R}^n$.

Vector Spaces 'V': a specific type of subspace

$u, v, w \in V$ and for all scalars $c, d \in \mathbb{R}$:

⇒ $\mathbf{u} + \mathbf{v}$ is in V . (V is "closed under addition".)

⇒ $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

⇒ $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

⇒ There is a vector (called the zero vector) $\mathbf{0}_V$ in V such that $\mathbf{u} + \mathbf{0}_V = \mathbf{u}$.

⇒ For each $\mathbf{u}$ in V , there is a vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}_V$.

⇒ $c\mathbf{u}$ is in V . (V is "closed under scalar multiplication".)

⇒ $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

⇒ $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

⇒ $(cd)\mathbf{u} = c(d\mathbf{u})$.

⇒ $1\mathbf{u} = \mathbf{u}$.

Column Spaces

Definition. The **column space**, written as $\text{Col}(A)$, of an $m \times n$ matrix A is the set of all linear combinations of the columns of A . If $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$, then $\text{Col}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$.

$$A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[\substack{R_2 - R_1 \rightarrow R_2 \\ R_3 + 2R_1 \rightarrow R_3}]{\text{}} \begin{bmatrix} 1 & -10 & -24 & -42 \\ 0 & \underline{2} & 6 & 10 \\ 0 & 0 & \underline{3} & 3 \end{bmatrix}$$

$$\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix}, \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \right\}$$

How to solve for $\text{Col}(A)$:

1. Put matrix A into REF
2. Find all the pivots of A
3. Map the pivots to the columns of your original matrix, A

Null Spaces

Definition. The **nullspace** of an $m \times n$ matrix A , written as $\text{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$; that is, $\text{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \mathbf{0}\}$.

How to solve for $\text{Nul}(A)$:

1. Set matrix A into **Augmented Matrix** with zeros on the right ($A\mathbf{x} = \mathbf{0}$)
2. Get A into **RREF**
3. Solve for $\mathbf{x}$

$$\text{Nul}(A) = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \dots)$$

Null Space Example

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} A = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bullet x_1 - 2x_2 - x_4 + 3x_5 = 0$$

$$\bullet x_3 + 2x_4 - 2x_5 = 0$$

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_3 = -2x_4 + 2x_5$$

$$\begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Linear Independence

Definition. Vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).

We say the vectors are **linearly dependent** if they are not linearly independent.

Theorem 30. Let A be an $m \times n$ matrix. The following are equivalent:

- ➡ The columns of A are linearly independent.
- ➡ $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.
- ➡ A has n pivots.
- ➡ there are no free variables for $A\mathbf{x} = \mathbf{0}$.

Basis and Dimension

Definition. Let V be a vector space. A sequence of vectors $(\mathbf{v}_1, \dots, \mathbf{v}_p)$ in V is a **basis** of V if

- ➡ $V = \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$, and
- ➡ $(\mathbf{v}_1, \dots, \mathbf{v}_p)$ are linearly independent.

The number of vectors in a basis of V is the **dimension** of V .

Basis and Dimension example

Is $\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right)$ a basis of $\mathbb{R}^3$?

Our set has 3 vectors and

$$\dim \mathbb{R}^3 = 3$$

General Definition of dimension

$$\dim \mathbb{R}^n = n$$

Next, we check linear independence

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

*Recall: All pivots for $n \times n$ matrix means **linear independence**

Theorem 33. A basis is a *minimal* spanning set of V ; that is the elements of the basis span V but you cannot delete any of these elements and still get all of V .

Basis and Dim of four subspaces:

Rank $[r]$: Number of pivots matrix has

Let A be an $m \times n$ matrix with rank r

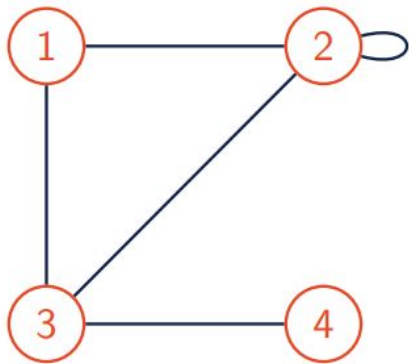
- $\dim \text{Nul}(A) = n - r$
- $\dim \text{Col}(A) = r$
- $\dim \text{Nul}(A^T) = m - r$
- $\dim \text{Col}(A^T) = r$

Graphs and Adjacency Matrices

A **graph** is a set of nodes (or: vertices) that are connected through edges.

Definition. Let $\mathcal{G}$ be a graph with n nodes. The **adjacency matrix** of $\mathcal{G}$ is the $n \times n$ -matrix $A = (a_{ij})$ such that

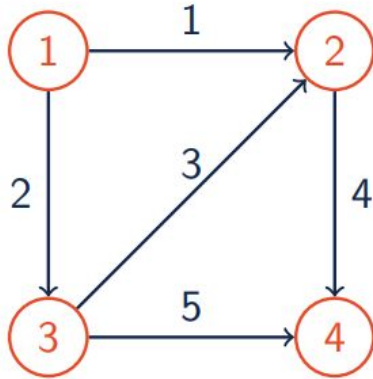
$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between node } i \text{ and node } j \\ 0 & \text{otherwise .} \end{cases}$$



N1	N2	N3	N4	
0	1	1	0	Node 1: Connected to N2 & N3
1	1	1	0	Node 2: Connected to N1, N2, & N3
1	1	0	1	Node 3: Connected to N1, N2 & N4
0	0	1	0	Node 4: Connected to N3

Directed Graphs

Definition. A **directed graph** is a set of vertices connected by edges, where the edges have a direction associated with them.



N1	N2	N3	N4	
0	0	0	0	Node 1: Nothing pointing to N1
1	0	1	0	Node 2: N1 and N3 pointing to N2
1	0	0	0	Node 3: N1 points to N3
0	1	1	0	Node 4: N2 and N3 pointing to N4

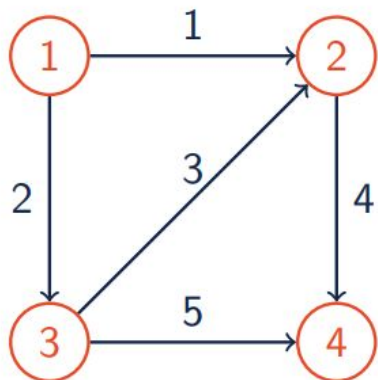
Definition. Let G be a directed graph with m edges and n nodes. The **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} 1, & \text{if there is a directed edge from node } j \text{ to node } i \\ 0, & \text{otherwise} \end{cases}$$

Edge-Node Incidence

Definition. Let G be a directed graph with m edges and n nodes. The **edge-node incidence matrix** of G is the $m \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$



	N1	N2	N3	N4	
$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$					Edge 1: Leaves N1; Enters N2
					Edge 2: Leaves N1; Enters N3
					Edge 3: Leaves N3; Enters N2
					Edge 4: Leaves N2; Enters N4
					Edge 5: Leaves N3; Enters N4

Orthogonal Complements

Definition. Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is the subspace $W^\perp$ of all vectors that are orthogonal to W ; that is

$$W^\perp := \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

Some helpful theorems:

- $(W^\perp)^\perp = W$
- $\text{Nul}(A) = \text{Col}(A^T)^\perp$
- $\text{Nul}(A)^\perp = \text{Col}(A^T)$
- $\text{Nul}(A^T) = \text{Col}(A)^\perp$

Theorem 43. Let V be a subspace of $\mathbb{R}^n$. Then $\dim V + \dim V^\perp = n$.

Coordinates

Standard basis (ε):

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Generally, if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are a basis B of vector space V , the coordinate vector of any vector $\mathbf{w}$ in V is:

$$\mathbf{w}_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \quad \text{if } \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

This coordinate vector is unique!

Coordinate Example

Let $V = \mathbb{R}^2$, and consider the bases

$$\mathcal{B} := \left(\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

$$\mathcal{E} := \left(\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \text{ Determine } \mathbf{w}_{\mathcal{B}} \text{ and } \mathbf{w}_{\mathcal{E}}$$

We want to find ' $\mathbf{w}$ ' in terms of $\mathbf{B}$'s and $\mathbf{E}$'s coordinate planes

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This is what $[3,-1]$ looks like in 'basis' $\mathbf{B}$

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_{\mathcal{E}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

This is what $[3,-1]$ looks like in 'basis' $\mathbf{E}$

Change of Basis Matrix

Definition. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of $\mathbb{R}^n$. The **change of basis matrix** $I_{\mathcal{C},\mathcal{B}}$ is the matrix such that for all $\mathbf{v} \in \mathbb{R}^n$

$$I_{\mathcal{C},\mathcal{B}}\mathbf{v}_{\mathcal{B}} = \mathbf{v}_{\mathcal{C}}$$

Matrix allowing us to go from coordinates **mapped in B** to be **mapped onto C**

Theorem 45. Let $\mathcal{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ be a basis of $\mathbb{R}^n$. Then

$$I_{\mathcal{E}_n,\mathcal{B}} = [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n]$$

That is, for all $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = [\mathbf{b}_1 \quad \dots \quad \mathbf{b}_n] \mathbf{v}_{\mathcal{B}}.$$

How do we compute change of basis matrix:

$$I_{B,C}?$$

What we know:

- $I_{E_n,B}$ = Matrix that maps **coordinates in B** onto **Standard**
- I_{B,E_n} = Matrix that maps **coordinates in Standard** onto **B**
- $I_{E_n,C}$ = Matrix that maps **coordinates in C** onto **Standard**
- I_{C,E_n} = Matrix that maps **coordinates in Standard** onto **C**

$$I_{B,E_n} I_{E_n,C}$$

From right to left:

We map coordinates **from C into the standard** coordinate plane, **then**, we map the newly acquired **standard coordinates onto B's coordinate plane**

AKA: $I_{B,C}$

Orthogonal and Orthonormal Bases

Definition. An **orthogonal basis** (an **orthonormal basis**) is an orthogonal set of vectors (an orthonormal set of vectors) that forms a basis.

Theorem 47. Let $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$ be an orthogonal basis of $\mathbb{R}^n$, and let $\mathbf{v} \in \mathbb{R}^n$. Then

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1} \mathbf{b}_1 + \dots + \frac{\mathbf{v} \cdot \mathbf{b}_n}{\mathbf{b}_n \cdot \mathbf{b}_n} \mathbf{b}_n.$$

When $\mathcal{B}$ is orthonormal, then $\mathbf{b}_i \cdot \mathbf{b}_i = 1$ for $i = 1, \dots, n$.

Theorem 48. Let $\mathcal{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ be an orthonormal basis of $\mathbb{R}^n$. Then

$$h_{\mathcal{U}, \mathcal{E}_n} = [\mathbf{u}_1 \quad \dots \quad \mathbf{u}_n]^T.$$

Why? An $n \times n$ -matrix Q is **orthogonal** if $Q^{-1} = Q^T$

In-Person Resources

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of Grainger Library!

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Course Office hours:

TAs: In-person: Mondays - Thursdays 5-7 pm

Focused on lectures and labs

CAs: On Zoom, schedule on Canvas.

Focused on Python

Instructors:

Chuang: Monday, 3-4pm, 233 CAB

Janda: Mondays 4-5pm, Harker Hall 304C

Baryshnikov: Tuesdays, 9:30-10:30 am

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	3pm-8pm	12pm-7pm	4pm-8pm	12pm-7pm	12pm-5pm 6pm-8pm	12pm- 4pm	2pm-6pm

Online Resources

Vector Spaces:

https://www.youtube.com/watch?v=XDvSsDsLVLs&ab_channel=TrevTutor

Linear Combinations, Spans, and Basis Vectors:

https://www.youtube.com/watch?v=k7RM-ot2NWy&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=2&ab_channel=3Blue1Brown

Change of Basis:

https://www.youtube.com/watch?v=P2LTAUO1TdA&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=13&ab_channel=3Blue1Brown

Abstract Vector Spaces:

https://www.youtube.com/watch?v=TgKwz5lkpc8&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=16&ab_channel=3Blue1Brown

Questions?



Join the queue to see the worksheet and slides!

Feedback form



Help us improve our sessions for the future!

Important Definitions

Rank - Nullity Theorem for $m \times n$ matrix: $\dim(\text{col}(A)) + \dim(\text{null}(A)) = n$

Linearity: $T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$

Dimension: $\dim(\text{col}(A)) = \text{rank} = \# \text{pivot vars.}$
 $\dim(\text{null}(A)) = \text{nullity} = \# \text{free vars.}$

Subspace Criterion: Closed under linear combination, defined as the span of basis vectors

Linear Independence: $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$
iff $x_1 = x_2 = \cdots = x_n = 0$

Orthogonal Complements: $\text{col}(A)^\perp = \text{null}(A^\top)$
 $\text{col}(A^\top)^\perp = \text{null}(A)$

Coordinate Inverse: $I_{\mathcal{A}\mathcal{E}} = I_{\mathcal{E}\mathcal{A}}^{-1}$