



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 231 — Engineering Calculus

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Tutors: September 29th, 7-9:00 PM Yash, Aadya, Aditya, Amishi

Can't make it to a session? Here's our schedule by course:

<https://care.grainger.illinois.edu/tutoring/schedule-by-subject>

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: <https://queue.illinois.edu/q/queue/979>
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Use differentials to approximate

$$\sqrt{24.8}$$

- (a) 4.80
- (b) 4.98
- (c) 5.02
- (d) 5.20

Let $f(x) = \sqrt{x}$. We choose $x = 25$ since $\sqrt{25} = 5$.

$$dx = 24.8 - 25 = -0.2$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$dy = f'(25)dx = \frac{1}{10}(-0.2) = -0.02$$

$$\sqrt{24.8} \approx 5 - 0.02 = 4.98$$

The answer is **(b)**.

2. A quantity $P(t)$ decays exponentially. Suppose

$$P(2) = 72 \quad \text{and} \quad P(6) = 18$$

What is $P(0)$?

- (a) 99
- (b) 108
- (c) 144
- (d) 288

From $t = 2$ to $t = 6$, the quantity changes by a factor of

$$\frac{P(6)}{P(2)} = \frac{18}{72} = \frac{1}{4}$$

This occurs over 4 units of time. That means over half that time interval, 2 units, the quantity would change by a factor of

$$\sqrt{\frac{1}{4}} = \frac{1}{2}$$

Going backwards from $t = 2$ to $t = 0$ reverses this change, so

$$\implies P(0) = 2P(2) = 2(72) = 144$$

The answer is **(c)**.

3. The surface area of a spherical balloon is decreasing at $24\pi \text{ cm}^2/\text{s}$. At the instant when the radius is 3 cm, how quickly is the volume of the balloon changing?

- (a) $-24\pi \text{ cm}^3/\text{s}$
- (b) $-36\pi \text{ cm}^3/\text{s}$
- (c) $36\pi \text{ cm}^3/\text{s}$
- (d) $-72\pi \text{ cm}^3/\text{s}$

The surface area of a sphere is

$$A = 4\pi r^2$$

Differentiating with respect to time,

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

Since the surface area is decreasing,

$$-24\pi = 8\pi(3) \frac{dr}{dt}$$

$$\frac{dr}{dt} = -1$$

We know that

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi(3)^2(-1) = -36\pi \text{ cm}^3/\text{s}$$

The answer is **(b)**.

4. Consider the curve

$$x^2 + xy + y^2 = 12$$

At which of the following points does the curve have a vertical tangent?

- (a) $(2, -4)$ and $(-2, 4)$
- (b) $(4, -2)$ and $(-4, 2)$
- (c) All four points above
- (d) None of the points above

Differentiating w.r.t x

$$2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0$$

Taking all the $\frac{dy}{dx}$ terms to the same side,

$$(x + 2y) \frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

A vertical tangent occurs when the denominator is zero:

$$x + 2y = 0$$

$$x = -2y$$

Substitute this into the original equation:

$$(-2y)^2 + (-2y)y + y^2 = 12$$

$$3y^2 = 12$$

$$y = \pm 2$$

That means, the points are

$$(4, -2) \text{ and } (-4, 2)$$

The answer is **(b)**.

5. If

$$y = (x^2 + 1)^{\sin(x)}$$

which expression equals y' ?

- (a) $(x^2 + 1)^{\sin(x)} \left[\cos(x) \ln(x^2 + 1) + \frac{2x \sin(x)}{x^2 + 1} \right]$
- (b) $(x^2 + 1)^{\sin(x)} \left[\cos(x) + \frac{2x \sin(x)}{x^2 + 1} \right]$
- (c) $(x^2 + 1)^{\sin(x)} \left[\sin(x) \ln(x^2 + 1) + \frac{2x \cos(x)}{x^2 + 1} \right]$
- (d) $\sin(x)(x^2 + 1)^{\sin(x)-1}(2x)$

Take the logarithm of both sides:

$$\ln(y) = \sin(x) \ln(x^2 + 1)$$

$$\frac{1}{y} \frac{dy}{dx} = \cos(x) \ln(x^2 + 1) + \sin(x) \frac{2x}{x^2 + 1}$$

$$\frac{dy}{dx} = y \left[\cos(x) \ln(x^2 + 1) + \frac{2x \sin(x)}{x^2 + 1} \right]$$

Since $y = (x^2 + 1)^{\sin(x)}$,

$$y' = (x^2 + 1)^{\sin(x)} \left[\cos(x) \ln(x^2 + 1) + \frac{2x \sin(x)}{x^2 + 1} \right]$$

The answer is **(a)**.

6. A 10 m ladder leans against a vertical wall. The bottom of the ladder moves away from the wall at 2 m/s. At an instant when the bottom is 6 m from the wall, how quickly is the area of the triangle formed by the wall, floor, and ladder changing?

- (a) $-\frac{7}{2}$ m²/s
(b) $\frac{7}{2}$ m²/s
(c) 8 m²/s
(d) 16 m²/s

Let x be the distance of the bottom of the ladder from the wall and y be the height of the ladder. Since the ladder has a fixed length of 10 m,

$$x^2 + y^2 = 100$$

When $x = 6$,

$$6^2 + y^2 = 100$$

$$y = 8$$

Differentiating the equation:

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

Substituting $x = 6$, $y = 8$, and $\frac{dx}{dt} = 2$,

$$2(6)(2) + 2(8) \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{3}{2}$$

We know that

$$A = \frac{1}{2}xy$$

$$\frac{dA}{dt} = \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right)$$

$$\frac{dA}{dt} = \frac{1}{2} \left[6 \left(-\frac{3}{2} \right) + 8(2) \right]$$

$$\frac{dA}{dt} = \boxed{\frac{7}{2} \text{ m}^2/\text{s}}$$

The answer is **(b)**.

7. Let

$$f(x) = x^2 - 6x + 11$$

Which statement is correct?

- (a) f has a local maximum of 2 at $x = 3$.
- (b) f has a local minimum of 2 at $x = 3$.
- (c) f has a local minimum of 3 at $x = 2$.
- (d) f has no local extrema.

Differentiating both sides,

$$f'(x) = 2x - 6$$

Since the slope and derivative are zero at a critical point,

$$2x - 6 = 0$$

$$x = 3$$

$$f''(x) = 2 > 0$$

Which means that $x = 3$ is a local minimum.

$$f(3) = 3^2 - 6(3) + 11 = 2$$

That means f has a local minimum of 2 at $x = 3$.

The answer is **(b)**.

8. Which definite integral is represented by

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(1 + \frac{3k}{n}\right)^2 \frac{3}{n}?$$

(a) $\int_1^4 x^2 dx$

(b) $\int_1^3 x^2 dx$

(c) $\int_0^4 x^2 dx$

(d) $\int_1^4 x^3 dx$

A Riemann sum has the form

$$\sum_{k=1}^n f(x_k) \Delta x$$

Here,

$$\Delta x = \frac{3}{n}$$

$$\implies b - a = 3$$

We also have

$$x_k = 1 + \frac{3k}{n}$$

$$a = 1 \implies b = 4$$

Since the function is,

$$f(x) = x^2$$

$$\implies \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(1 + \frac{3k}{n}\right)^2 \frac{3}{n} = \int_1^4 x^2 dx$$

The answer is **(a)**.

9. A function f is approximated on $[0, 8]$ using 4 equal subintervals. Suppose

$$L_4 = 14 \quad \text{and} \quad R_4 = 22$$

What is

$$f(8) - f(0)?$$

- (a) 2
- (b) 4
- (c) 8
- (d) 16

The width of each subinterval is

$$\Delta x = \frac{8 - 0}{4} = 2$$

$$\implies L_4 = 2[f(0) + f(2) + f(4) + f(6)]$$

$$R_4 = 2[f(2) + f(4) + f(6) + f(8)]$$

Subtracting the two equations gives

$$R_4 - L_4 = 2[f(8) - f(0)]$$

$$\implies 22 - 14 = 2[f(8) - f(0)]$$

$$8 = 2[f(8) - f(0)]$$

$$\boxed{f(8) - f(0) = 4}$$

The answer is **(b)**.

10. Suppose $f(x) \geq 0$ on $[0, 2]$ and $f(x) \leq 0$ on $[2, 5]$, with

$$\int_0^2 f(x) dx = 5$$

and

$$\int_2^5 f(x) dx = -3$$

Which pair gives

$$\left(\int_0^5 f(x) dx, \text{ total geometric area between } f \text{ and the } x\text{-axis} \right)?$$

- (a) (2, 2)
- (b) (8, 8)
- (c) (2, 8)
- (d) (8, 2)

A definite integral gives signed area.

$$\begin{aligned} \Rightarrow \int_0^5 f(x) dx &= \int_0^2 f(x) dx + \int_2^5 f(x) dx \\ &= 5 + (-3) = 2 \end{aligned}$$

Geometric area counts both regions positively, so

$$5 + 3 = 8$$

$\Rightarrow \text{ the pair is } (2, 8)$

The answer is **(c)**.

11. Define

$$F(x) = \int_2^x (t^3 - 4t + 1) dt$$

What is $F'(x)$?

- (a) $\frac{x^4}{4} - 2x^2 + x$
- (b) $x^3 - 4x + 1$
- (c) $x^3 - 4x$
- (d) 1

By the Fundamental Theorem of Calculus,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$
$$\implies \boxed{F'(x) = x^3 - 4x + 1}$$

The answer is **(b)**.

12. Consider

$$\int \frac{x^9}{(1+x^2)^6} dx$$

Which substitution leads most directly to evaluating the integral?

- (a) $u = 1 + x^2$
- (b) $u = 1 + \frac{1}{x^2}$
- (c) $u = x^9$
- (d) $u = (1 + x^2)^6$

First, rewrite

$$1 + x^2 = x^2 \left(1 + \frac{1}{x^2} \right)$$
$$\implies (1 + x^2)^6 = x^{12} \left(1 + \frac{1}{x^2} \right)^6$$
$$\implies \int \frac{x^9}{x^{12} \left(1 + \frac{1}{x^2} \right)^6} dx$$

$$= \int \frac{1/x^3}{(1 + \frac{1}{x^2})^6} dx$$

Let

$$u = 1 + \frac{1}{x^2}$$

$$du = -\frac{2}{x^3} dx$$

$$\implies \frac{1}{x^3} dx = -\frac{1}{2} du$$

$$\implies -\frac{1}{2} \int u^{-6} du$$

That means that the most direct substitution is

$$\boxed{u = 1 + \frac{1}{x^2}}$$

The answer is **(b)**.

13. For $x > 0$, evaluate

$$\int (\ln x)^2 dx$$

- (a) $x(\ln x)^2 - 2x \ln x + 2x + C$
- (b) $x(\ln x)^2 - 2x \ln x + C$
- (c) $x(\ln x)^2 + 2x \ln x - 2x + C$
- (d) $\frac{(\ln x)^3}{3} + C$

Use integration by parts with

$$u = (\ln x)^2 \quad dv = dx$$

$$du = \frac{2 \ln x}{x} dx \quad v = x$$

$$\implies \int (\ln x)^2 dx = x(\ln x)^2 - 2 \int \ln x dx$$

Use integration by parts again on

$$\int \ln x dx$$

$$u = \ln x \quad dv = dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$\int \ln x dx = x \ln x - x$$

$$\implies \int (\ln x)^2 dx = x(\ln x)^2 - 2(x \ln x - x)$$

$$\boxed{x(\ln x)^2 - 2x \ln x + 2x + C}$$

The answer is **(a)**.

14. When applying integration by parts to

$$\int x^3 e^x dx$$

which choice is most appropriate according to LIATE?

- (a) $u = x^3, dv = e^x dx$
- (b) $u = e^x, dv = x^3 dx$
- (c) $u = x^3 e^x, dv = dx$
- (d) Integration by parts should not be used.

According to LIATE, an algebraic function should be chosen as u before an exponential function.

$$\implies \boxed{u = x^3 \quad dv = e^x dx}$$

$$du = 3x^2 dx \quad v = e^x$$

The answer is **(a)**.