



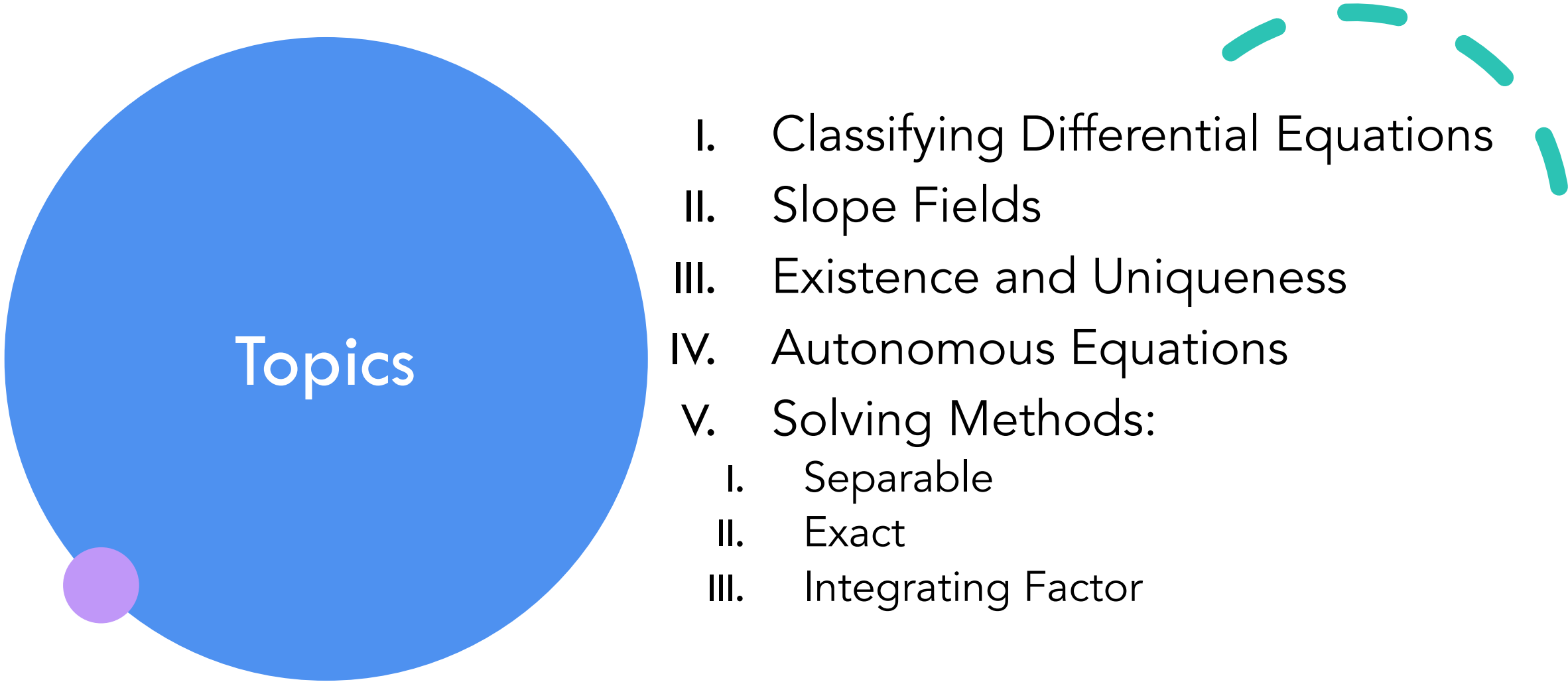
MATH 285

Midterm 1 Review

CARE

Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.



Topics

- I. Classifying Differential Equations
- II. Slope Fields
- III. Existence and Uniqueness
- IV. Autonomous Equations
- V. Solving Methods:
 - I. Separable
 - II. Exact
 - III. Integrating Factor

Differential Equations

- "A differential equation is any relationship between a function (usually denoted $y(t)$) and its derivatives up to some order."
- Slope Fields: Help visually model a differential equation
 - Lines parallel to the derivative at each point
 - Can show overall direction and shape of the solution, as well as equilibrium values

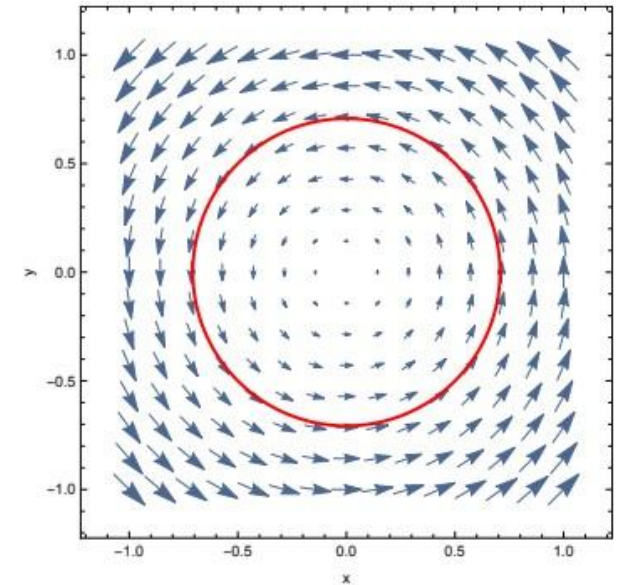


Figure 1.4: A slope field for $\frac{dy}{dt} = -\frac{t}{y}$ (blue) together with a solution curve (red).

Differential Equations. Bronski J., Manfroi A., Figure 1.4

Classifications

2nd order  Linear Ordinary

$$\frac{d^2y}{dt^2} + \sin(t) \frac{dy}{dt} + 15y = e^t$$

Ordinary vs Partial

- ODE's involve only standard derivatives
- PDE's involve partial derivatives

Linear vs Nonlinear

- Linear differential equations only have linear terms of the function and its derivatives
- Nonlinear equations are everything else

Order

- The order of a differential equation is the degree of the highest derivative it contains

Existence and Uniqueness Theorem

$$\frac{dy}{dt} = f(y, t) \qquad y(t_0) = y_0$$

- A solution to the differential equation is **guaranteed to exist** in the **interval** in which the **first derivative is continuous around the initial value**
- That solution is **guaranteed to be** unique if $\frac{\partial f(y, t)}{\partial y}$ is also **continuous around the initial value**

Autonomous Equations

- Autonomous equation: **does not explicitly involve independent variable**

$$\frac{dy}{dt} = f(y)$$

- Draw a **phase line**, identify points where the **derivative is 0**, and then **identify equilibria**
- Types of equilibria:
 - **Stable**: nearby points converge to the equilibrium
 - **Semi-stable**: points converge from one direction
 - **Unstable**: points diverge away from the equilibrium

Separable Equations

- Separable Equations can be written as:

$$\frac{dy}{dt} = f(y)g(t)$$

- If your equation is separable, it can be solved **directly through integration**:

$$\int \frac{dy}{f(y)} = \int g(t)dt + C$$

Exact Equations

- Exact equations have the form of:

$$N(x, y) \cdot y' + M(x, y) = 0$$

- An equation is exact if the **partial derivatives of the two coefficient terms are equal**:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Solving Exact Equations

1. **Partially integrate** either N or M :

$$\int N(x, y) dy$$

$$\int M(x, y) dx$$

2. **Set equal to Ψ** + a constant of integration function:

$$\Psi(x, y) = \int M(x, y) dx + g(y)$$

3. Take the **derivative** with respect to the **opposite variable**:

$$\Psi_y(x, y) = \frac{d}{dy} \int M(x, y) dx + g'(y)$$

4. **Set equal** to the **other term** you didn't integrate:

$$\Psi_x(x, y) = M(x, y)$$

$$\Psi_y(x, y) = N(x, y)$$

5. **Integrate** to solve for $f(x)$ or $f(y)$ and plug back into step 2

Exact Equation Example

Solve the following differential equation:

$$(5x^2y + 2x + 4)\frac{dy}{dt} + (5xy^2 + 2y + 7) = 0$$

Exact Equation Example

The two terms are:

$$M(x, y) = 5xy^2 + 2y + 7$$

$$N(x, y) = 5x^2y + 2x + 4$$

In this example, we choose M to integrate (w.r.t x)

$$\psi(x, y) = \int (5xy^2 + 2y + 7) dx$$

$$\psi(x, y) = \frac{5}{2}x^2y^2 + 2xy + 7x + h(y)$$

Exact Equation Example

Then differentiate with respect to y :

$$\psi_y = 5x^2y + 2x + h'(y)$$

And set this equal to N :

$$5x^2y + 2x + h'(y) = 5x^2y + 2x + 4$$

$$h'(y) = 4$$

$$h(y) = 4y$$

Exact Equation Example

Final Answer:

$$\frac{5}{2}x^2y^2 + 2xy + 7x + 4y = C$$

Integrating Factor Method

1. Make sure your equation looks like:

$$\frac{dy}{dt} + P(t)y = Q(t)$$

2. Calculate the **integrating factor**:

$$\mu(t) = e^{\int P(t)dt}$$

3. **Multiply the entire equation** by the integrating factor:

$$e^{\int P(t)dt} \left(\frac{dy}{dt} + P(t)y \right) = Q(t)e^{\int P(t)dt}$$

4. Re-write the left-hand side as the **result of product rule**:

$$\frac{d}{dt}(ye^{\int P(t)dt}) = Q(t)e^{\int P(t)dt}$$

5. **Integrate both sides** and rearrange to solve for $y(t)$:

$$y(t) = e^{-\int P(t)dt} \int Q(t)e^{\int P(t)dt} + Ce^{-\int P(t)dt}$$

Integrating Factor Example

Solve the following differential equation:

$$y' + 3y = 2$$

Integrating Factor Example

The integrating factor is:

$$\mu(x) = e^{\int P(x) dx}$$

$$\mu(x) = e^{\int 3 dx} = e^{3x}$$

So multiply both sides by the integrating factor:

$$e^{3x}y' + 3e^{3x}y = 2e^{3x}$$

The left side then becomes:

$$\frac{d}{dx}(e^{3x}y)$$

Integrating Factor Example

Then integrate both sides:

$$\frac{d}{dx}(e^{3x}y) = 2e^{3x}$$

$$e^{3x}y = \int 2e^{3x} dx$$

$$e^{3x}y = \frac{2}{3}e^{3x} + C$$

$$y = \frac{2}{3} + Ce^{-3x}$$

And solve for y:

$$y = \frac{2}{3} + Ce^{-3x}$$

Integrating Factor Example

Final Answer:

$$y = \frac{2}{3} + Ce^{-3x}$$

Substitutions

- Similar to u-substitutions from calculus
- Find a variable $v(y,t)$
- Find dy/dt in terms of v
- Plug v terms into our equation
- Replace v with y and t

Substitutions Special Cases

Scale Invariant Form

- We can write $v = y/t$

$$\frac{dy}{dt} = F\left(\frac{y}{t}\right)$$

$$v = \frac{y}{t}, \quad y = vt$$

$$\frac{dy}{dt} = t \frac{dv}{dt} + v = F(v)$$

Reducible

- Occurs when there is no y term, so we can say $v = y'$
- Thus $dv/dt = y''$
- Occurs when there is no t term
- We can set $y' = p(y)$ and $y'' = y'(dp/dy) = p(y)(dp/dy)$

Bernoulli Equation

Equations of this form:

$$\frac{dy}{dt} + P(t)y = Q(t)y^a$$

$$y^{-a} \frac{dy}{dt} + P(t)y^{1-a} = Q(t)$$

$$w = y^{1-a}, \quad \frac{dw}{dt} = (1-a)y^{-a} \frac{dy}{dt}$$

$$\frac{1}{1-a} \frac{dw}{dt} + P(t)w = Q(t)$$

Substitution Example

$$\frac{dy}{dt} = (3y + t)t^2 - \frac{1}{3}$$

Substitution Example

Find a suitable v and write dy/dt in terms of v :

$$v = 3y + t$$

$$\frac{dv}{dt} = 3\frac{dy}{dt} + 1$$

$$\frac{dy}{dt} = \frac{1}{3}\left(\frac{dv}{dt} - 1\right)$$

Substitution Example

Plug v into our original equation:

$$\frac{1}{3} \left(\frac{dv}{dt} - 1 \right) = vt^2 - \frac{1}{3}$$

$$\frac{dv}{dt} - 1 = 3vt^2 - 1$$

$$\frac{dv}{dt} = 3vt^2$$

Substitution Example

Integrate both sides and solve for v:

$$\int \frac{1}{v} dv = \int 3t^2 dt$$

$$\ln(v) = t^3 + C$$

$$v = Ce^{t^3}$$

Substitution Example

Use v to solve for y :

$$y = \frac{v - t}{3}$$

$$y = \frac{Ce^{t^3} - t}{3}$$

Thanks for
Coming!

