



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 231 E — Engineering Calculus

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Tutors: Yash, Aadya, Aditya, Amishi

Can't make it to a session? Here's our schedule by course:

<https://care.grainger.illinois.edu/tutoring/schedule-by-subject>

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: <https://queue.illinois.edu/q/queue/979>
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Which of the following are equivalent to the following?

$$\frac{1}{4+i}$$

- (a) $4 - i$
 (b) $i + 4$
 (c) $\frac{4}{17} - \frac{i}{17}$
 (d) $\frac{4}{\sqrt{17}} - \frac{i}{\sqrt{17}}$

To start off, we need to multiply by the conjugate of

$$\frac{1}{4+i}$$

$$\frac{1}{4+i} * \frac{4-i}{4-i} = \frac{4-i}{16+1}$$

$$\frac{4-i}{16+1} = \boxed{\frac{4}{17} - \frac{i}{17}}$$

2. What is the simplified form of the following?

$$\left(\frac{2+i}{1+4i}\right)^2$$

- (a) $\frac{3}{8i-15}$
 (b) $\frac{24}{5} + \frac{7}{5}i$
 (c) $-\frac{20}{46} - \frac{30}{46}i$
 (d) $\frac{-13}{289} - \frac{84}{289}i$

To start solving this problem, we need to multiply by the conjugate.

$$\frac{2+i}{1+4i} * \frac{1-4i}{1-4i} = \frac{6-7i}{17}$$

But we still need to square this value!

$$\left(\frac{6-7i}{17}\right)^2 = \frac{36-84i-49}{289}$$

$$\frac{36 - 84i - 49}{289} = \boxed{\frac{-13}{289} - \frac{84i}{289}}$$

3. Compute the value of

$$\frac{(-1 + i\sqrt{3})^{2026}}{(1 + i)^{2020}}$$

- (a) $2^{1016}e^{i\frac{5\pi}{3}}$
- (b) $2^{1016}e^{i\frac{\pi}{3}}$
- (c) $2^{1010}e^{i\frac{5\pi}{3}}$
- (d) $2^{2026}e^{i\frac{5\pi}{3}}$

First, convert the complex numbers into polar form.

For $-1 + i\sqrt{3}$,

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$$

Since the point $(-1, \sqrt{3})$ lies in Quadrant II and has angle $\frac{\pi}{3}$,

$$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\implies -1 + i\sqrt{3} = 2e^{i\frac{2\pi}{3}}$$

Similarly, $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$

Now, substitute these into the original expression

$$\begin{aligned} \frac{(-1 + i\sqrt{3})^{2026}}{(1 + i)^{2020}} &= \frac{\left(2e^{i\frac{2\pi}{3}}\right)^{2026}}{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{2020}} \\ &= \frac{2^{2026}}{(\sqrt{2})^{2020}} e^{i\left(2026 \cdot \frac{2\pi}{3} - 2020 \cdot \frac{\pi}{4}\right)} \end{aligned}$$

Simplify the magnitude

$$\frac{2^{2026}}{2^{1010}} = 2^{1016}$$

Now, simplify the angle

$$\begin{aligned}
 2026 * \frac{2\pi}{3} - 2020 * \frac{\pi}{4} &= \frac{4052\pi}{3} - 505\pi \\
 &= \frac{2537\pi}{3}
 \end{aligned}$$

Since angles that differ by a full rotation of 2π represent the same point in the complex plane, we can subtract whole multiples of 2π until the angle is between 0 and 2π .

We choose 422 full rotations because it is the largest whole number of full rotations we can subtract without going below 0.

$$\begin{aligned}
 422(2\pi) &= 844\pi = \frac{2532\pi}{3} \\
 \frac{2537\pi}{3} - \frac{2532\pi}{3} &= \frac{5\pi}{3} \\
 \Rightarrow \boxed{\frac{(-1 + i\sqrt{3})^{2026}}{(1 + i)^{2020}} = 2^{1016} e^{i\frac{5\pi}{3}}}
 \end{aligned}$$

4. Compute the Limit:

$$\lim_{x \rightarrow \infty} \frac{2x + \sin(x)}{x}$$

- (a) 2
- (b) $\frac{1}{4}$
- (c) ∞
- (d) 0

$$\lim_{x \rightarrow \infty} \frac{2x + \sin(x)}{x} = \lim_{x \rightarrow \infty} 2 + \frac{\sin(x)}{x}$$

$$\lim_{x \rightarrow \infty} 2 + \frac{\sin(x)}{x} = 2 + \lim_{x \rightarrow \infty} \frac{\sin(x)}{x}$$

The rest of this problem can be handled with the squeeze theorem

$$\lim_{x \rightarrow \infty} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0^+} x \sin\left(\frac{1}{x}\right)$$

For $x > 0$

$$-x < x \sin\left(\frac{1}{x}\right) < x$$

$$-1 < \sin\left(\frac{1}{x}\right) < 1$$

$$\lim_{x \rightarrow 0^+} x = 0$$

So the Squeeze Theorem tells us this limit equals zero.

BUT! We still have a value of 2 from earlier so this limit equals $\boxed{2}$

5. Compute the Limit:

$$\lim_{x \rightarrow \infty} \frac{9x^{6968} + x^{6967} \sin(5x) + 27x^{1789}}{45x^{6968} + 54x^{1789}}$$

- (a) $\frac{1}{5}$
- (b) 0
- (c) $\frac{1}{3}$
- (d) does not exist

First, divide both the numerator and denominator by the highest power, x^{6968}

$$\lim_{x \rightarrow \infty} \frac{9x^{6968} + x^{6967} \sin(5x) + 27x^{1789}}{45x^{6968} + 54x^{1789}} = \lim_{x \rightarrow \infty} \frac{9 + \frac{\sin(5x)}{x} + 27x^{-5179}}{45 + 54x^{-5179}}$$

We know that

$$-1 \leq \sin(5x) \leq 1$$

$$-\frac{1}{x} \leq \frac{\sin(5x)}{x} \leq \frac{1}{x}$$

As $x \rightarrow \infty$,

$$-\frac{1}{x} \rightarrow 0 \quad \text{and} \quad \frac{1}{x} \rightarrow 0$$

So, by the Squeeze Theorem,

$$\lim_{x \rightarrow \infty} \frac{\sin(5x)}{x} = 0$$

$$\implies \lim_{x \rightarrow \infty} \frac{9 + \frac{\sin(5x)}{x} + 27x^{-5179}}{45 + 54x^{-5179}} = \frac{9 + 0 + 0}{45 + 0} = \boxed{\frac{1}{5}}$$

6. Compute the Limit:

$$\lim_{x \rightarrow 0} \frac{\sin(x^2) - x^2 + \frac{x^6}{6}}{x^{10}}$$

- (a) 0
- (b) $\frac{1}{120}$
- (c) $-\frac{1}{120}$
- (d) $\frac{1}{6}$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \dots$$

Now, substitute this into the limit

$$\lim_{x \rightarrow 0} \frac{\left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \dots\right) - x^2 + \frac{x^6}{6}}{x^{10}}$$

$$\lim_{x \rightarrow 0} \frac{\frac{x^{10}}{120} - \dots}{x^{10}}$$

Divide everything by x^{10}

$$\lim_{x \rightarrow 0} \left(\frac{1}{120} - \dots \right)$$

As $x \rightarrow 0$, all the remaining terms go to 0

$$\boxed{\frac{1}{120}}$$

7. Compute the Limit:

$$\lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{x^2}$$

- (a) 0
- (b) $\frac{3}{2}$
- (c) $\frac{9}{2}$

(d) does not exist

Direct substitution gives $\frac{0}{0}$, so we need to simplify using the half angle formula

$$\cos(x) = 1 - 2 \sin^2\left(\frac{x}{2}\right)$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{1 - \cos(3x)}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2\left(\frac{3x}{2}\right)}{x^2}$$

Rewrite the expression so that we can use the standard limit

$$\begin{aligned} \lim_{u \rightarrow 0} \frac{\sin(u)}{u} &= 1 \\ &= 2 \left(\frac{3}{2}\right)^2 \lim_{x \rightarrow 0} \left(\frac{\sin\left(\frac{3x}{2}\right)}{\frac{3x}{2}}\right)^2 \\ &= 2 \left(\frac{9}{4}\right) (1)^2 = \boxed{\frac{9}{2}} \end{aligned}$$

8. If $f(y) = \frac{y^2 \ln(y)}{1+y}$, compute $f'(1)$.

- (a) $\frac{1}{2}$
- (b) $\frac{1}{4}$
- (c) 1
- (d) 0

We will need to use the product, quotient, and chain rules for this problem

$$\begin{aligned} \frac{d}{dy} \left(\frac{y^2 \ln(y)}{1+y} \right) &= \frac{(1+y) \left(2y \ln(y) + y^2 * \frac{1}{y} \right) - y^2 \ln(y)}{(1+y)^2} \\ &= \frac{(1+y)(2y \ln(y) + y) - y^2 \ln(y)}{(1+y)^2} \\ f'(1) &= \frac{(2)(0+1) - 0}{(2)^2} = \boxed{\frac{1}{2}} \end{aligned}$$

9. Compute $f'(x)$ where $f(x) = x \cos(x^4)$

- (a) $-4x^4 \sin(x^4)$
- (b) $\sin(x^4) - 4x^4 \cos(x^4)$
- (c) $\cos(x^4) - 4x^4 \sin(x^4)$
- (d) $\cos(x^4)$

We will need to use the product and chain rules for this problem

$$\frac{d}{dx}(x \cos(x^4)) = \cos(x^4) * 1 + x\left(\frac{d}{dx}(\cos(x^4))\right)$$

$$\frac{d}{dx}(x \cos(x^4)) = \cos(x^4) + x(-4x^3 \sin(x^4))$$

$$\frac{d}{dx}(x \cos(x^4)) = \boxed{\cos(x^4) - 4x^4 \sin(x^4)}$$

10. If $f(x) = \left((x^2 + 1)^3 + 4x\right)^2$, then find $f'(1)$.

- (a) 144
- (b) 672
- (c) 512
- (d) 48

$$\frac{d}{dx}((x^2 + 1)^3 + 4x)^2 = 2((x^2 + 1)^3 + 4x)(3(x^2 + 1)^2(2x) + 4)$$

$$f'(x) = 2((x^2 + 1)^3 + 4x)(6x(x^2 + 1)^2 + 4)$$

Now, substitute 1 in for x

$$f'(1) = 2((1^2 + 1)^3 + 4(1))(6(1)(1^2 + 1)^2 + 4)$$

$$f'(1) = 2(12)(28) = 672$$

$$f'(1) = \boxed{672}$$

11. The Maclaurin series expansion of $e^{\sin x}$ is:

(a) $1 + x - \frac{x^2}{2} + \frac{x^4}{12} - \dots$

(b) $1 - x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$

(c) $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$

(d) $1 + x + \frac{x^2}{2} - \frac{x^4}{12} + \dots$

Let $f(x) = e^{\sin x}$.

$$f'(x) = e^{\sin x} \cos x = f(x) \cos x$$

$$f''(x) = f'(x) \cos x - f(x) \sin x$$

$$\begin{aligned} f'''(x) &= f''(x) \cos x - f'(x) \sin x - (f(x) \cos x + f'(x) \sin x) \\ &= f''(x) \cos x - 2f'(x) \sin x - f'(x) \end{aligned}$$

$$\begin{aligned} f^4(x) &= f'''(x) \cos x - f''(x) \sin x - 2(f'(x) \cos x + f''(x) \sin x) - f''(x) \\ &= f'''(x) \cos x - 3f''(x) \sin x - 2f'(x) \cos x - f''(x) \end{aligned}$$

Calculating higher order derivatives at $x = 0$,

$$f(0) = 1$$

$$f'(0) = 1 \times 1 = 1$$

$$f''(0) = 1 - 0 = 1$$

$$f'''(0) = 1 - 0 - 1 = 0$$

$$f^4(0) = 0 - 0 - 2 - 1 = -3$$

$$\begin{aligned} f^5(0) &= f^4(0) \cos 0 - 3f''(0) \cos 0 - 2f''(0) \cos 0 - f''(0) \\ &= -3 - 3 - 2 - 0 = -8 \end{aligned}$$

Using Maclaurin's expansion for infinite series,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) +$$

$$\frac{x^4}{4!}f^4(0) + \frac{x^5}{5!}f^5(0) + \frac{x^6}{6!}f^6(0) + \dots$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2!} \times 1 + 0 + \frac{x^4}{4!} \times (-3) + \frac{x^5}{5!} \times (-8) \dots$$

$$= \boxed{1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} \dots}$$