

Exam 1 Review Session

Math 231E



Please join the queue
for attendance!



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Outline

1. Please join the queue
2. Mini review of some topics covered
3. Practice! → CARE Worksheet, Practice Exams
 - a. Please raise hands for questions rather than put them in the queue



Need extra help? → 4th Floor Grainger Library

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 231 (E)	6pm-9pm	12pm-2pm	3pm-8pm			1pm-3pm	



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Content Review



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Complex Numbers

- Numbers of the form: $z = x + yi$
- Modulus: $|z| = \sqrt{x^2 + y^2}$
- Operations:
 - Addition: $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$
 - Multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$ (think of FOIL)
 - Division: $\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{x_2^2 + y_2^2}$
- Euler's Theorem: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$



Taylor Series

- Taylor series help approximate complicated functions into polynomials that are easier to evaluate

$$f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \dots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

- a is the value our Taylor series is evaluated at
 - Maclaurin series $\rightarrow a = 0$
- Big O Notation: indicates which term of the Taylor series is being “cut-off”
 - If we evaluate around a certain value, then any term put into the Big O Notation is insignificant to the polynomial overall



Common Functions to Use Taylor Series

<u>Function</u>	<u>Taylor Series</u>
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$
$\cos(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots + \frac{x^{2n}}{(2n)!}$
$\sin(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots + \frac{x^{2n+1}}{(2n+1)!}$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots + x^n$
$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^n \frac{x^{n+1}}{(n+1)}$

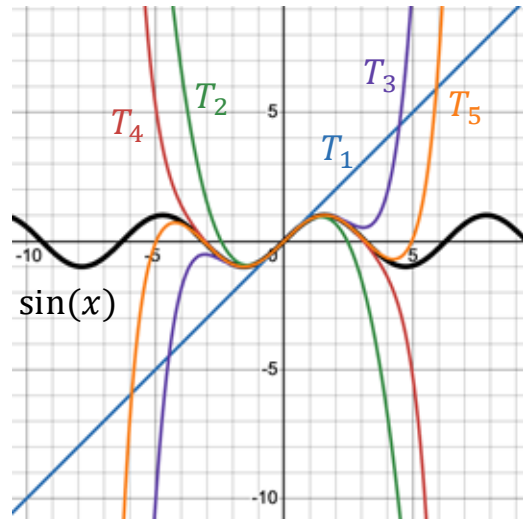


Uses and Applications of Taylor Series

1) Deriving other Taylor Series from the most common

- a) Substitution
- b) Derivatives
- c) Integrals

$$f(x) = \sin(x)$$



1) Evaluating limits with Taylor Polynomials versus the initial functions

1) Determining convergence or divergence of integrals and series → [later in the course](#)



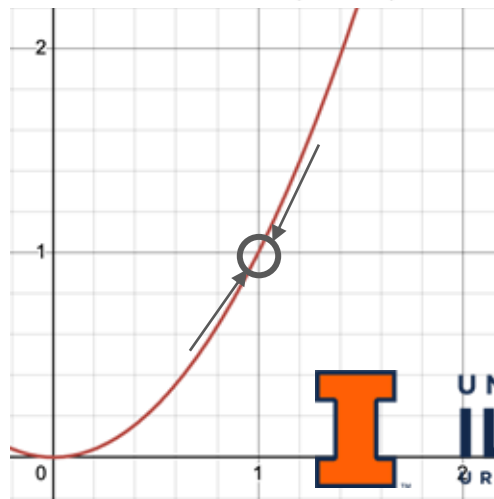
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Limits

- As we approach closer and closer to some value “a” from both the left and right hand side, the function gets closer and closer to “L”:

$$\lim_{x \rightarrow a} f(x) = L$$

- Epsilon-delta definition: For any $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.
 - $a \rightarrow$ “target input”
 - Delta $\rightarrow$ “allowable deviation from the target input”
 - Epsilon $\rightarrow$ “output tolerance”
 - $L \rightarrow$ “target output value”
- Infinite Limits: $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right)$



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Limit Laws

Operations

- **Addition:** $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- **Subtraction:** $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- **Multiplication:** $\lim_{x \rightarrow a} (f(x)g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
- **Division:** $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ (given the limit of the denominator is not 0)
- **Scaling by a constant:** $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$
- **Exponentiating:** $\lim_{x \rightarrow a} (f(x)^n) = \left(\lim_{x \rightarrow a} f(x) \right)^n$

Functions

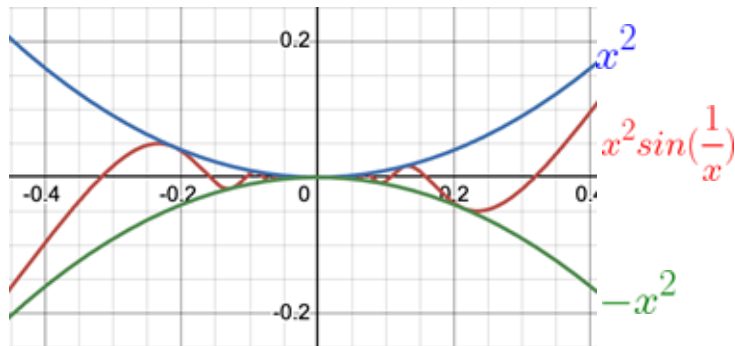
- **Constant:** $\lim_{x \rightarrow a} c = c$
- **Linear:** $\lim_{x \rightarrow a} x = a$
- **Power:** $\lim_{x \rightarrow a} x^n = a^n$
- **Root:** $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$

These laws can be combined to make finding limits easier!



Squeeze Theorem

- We have three functions such that near x : $f(x) \leq g(x) \leq h(x)$
- If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} g(x) = L$
- Great to use for functions that are hard to evaluate with limit laws



Derivatives

- Interpretation: the derivative of a function $f(x)$ at a represents the instantaneous rate of change at a

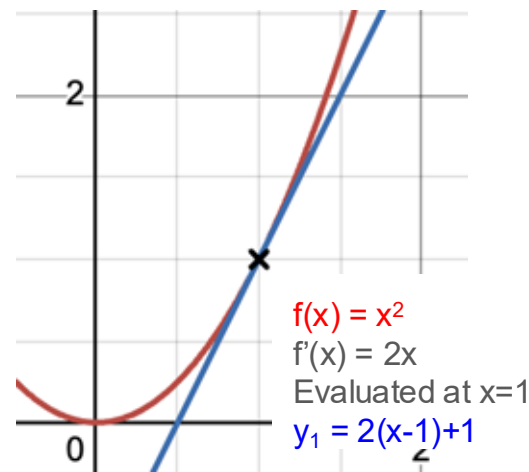
- Slope of tangent line at a : $y_a = f'(a)(x - a) + f(a)$

- Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$

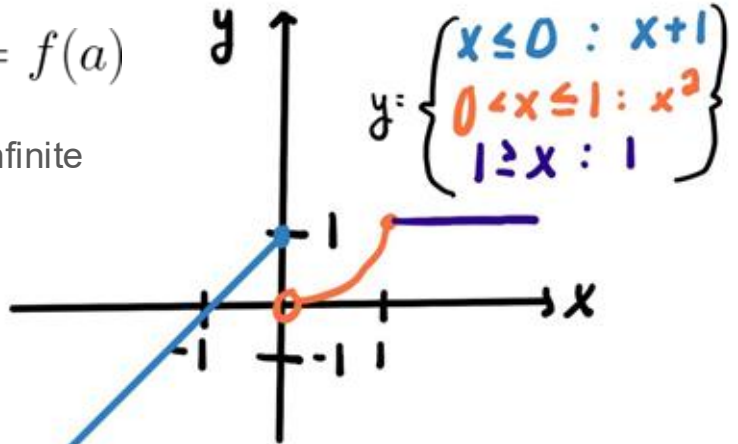
- Product Rule: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

- Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

- **Chain Rule:** $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$ (unwrap the layers of the function)



Continuity, Discontinuities, and Intermediate Value Theorem

- A function is continuous at a if $\lim_{x \rightarrow a} f(x) = f(a)$
 - Types of discontinuities: removable/point, jump, infinite
- 
- The graph shows a piecewise function on a Cartesian coordinate system. The x-axis has tick marks at -1, 0, and 1. The y-axis has tick marks at -1, 0, and 1. The function is defined by three segments: a blue line for $x \leq 0$ ending at a solid blue dot at (0, 1); an orange curve for $0 < x \leq 1$ starting at an open orange circle at (0, 0) and ending at a solid orange dot at (1, 1); and a purple horizontal line for $x \geq 1$ starting at a solid purple dot at (1, 1) and extending to the right. Handwritten notes in blue, orange, and purple ink define the segments: $y = \begin{cases} x \leq 0 : x+1 \\ 0 < x \leq 1 : x^2 \\ x \geq 1 : 1 \end{cases}$.



Mean Value Theorem

If:

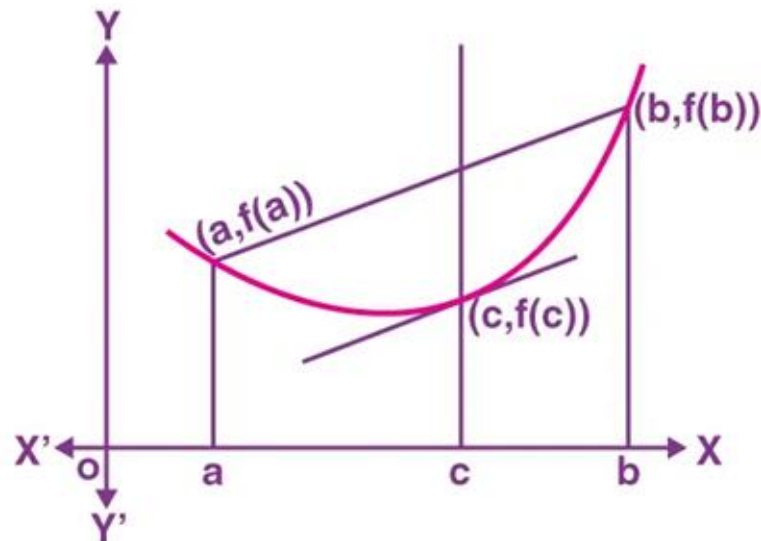
1. $f(x)$ is continuous $[a,b]$
2. $f(x)$ is differentiable (a,b)

Then:

There is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

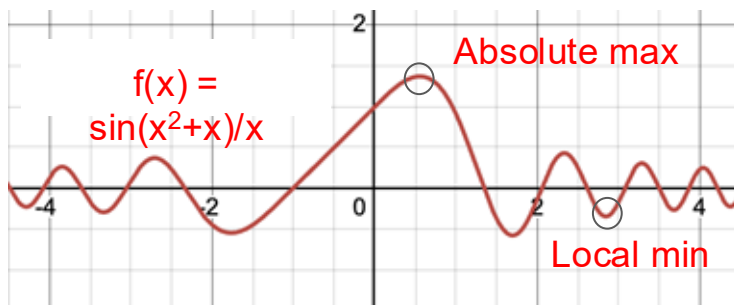
$$f(b) - f(a) = f'(c)(b - a)$$



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Minima and Maxima

- Absolute minima/maxima: smallest/largest outputs a function can produce on an interval
- Local minima/maxima: smallest/largest outputs of a function around a certain point

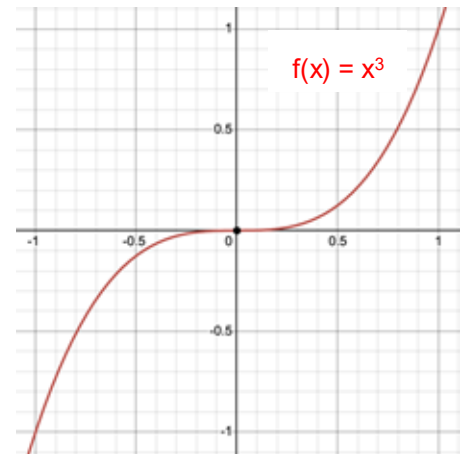


- Extreme Value Theorem: If $f(x)$ is continuous on $[a,b]$, then f achieves both an **absolute maximum and absolute minimum** on $[a,b]$
- Fermat's Theorem: If $f(x)$ has a **local minimum/maximum** at c and f' exists at c , then **$f'(c) = 0$**
 - We call c a **critical point**



Finding Extreme Values

1. Take the first derivative and set it equal to zero
 - a. Solve for all the critical points



1. Plug each critical point back into $f(x)$ and see which is the largest/smallest

-or-

2. Take the second derivative
 - a. If $f''(c)$ is $> 0 \rightarrow$ function is concave up $\rightarrow$ local minimum
 - b. If $f''(c)$ is $< 0 \rightarrow$ function is concave down $\rightarrow$ local maximum
 - c. If $f''(c) = 0 \rightarrow$ inconclusive

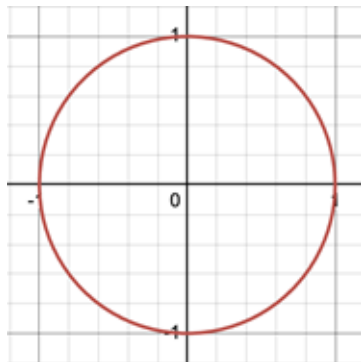


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Implicit Differentiation and Differentials

Implicit Differentiation

- Used when functions are defined in terms of both x and y
- Take the derivative of everything with respect to x
 - What is $\frac{dx}{dx}$? What is $\frac{dy}{dx}$?



Differentials

- For a small change in input Δx , the function will change proportionally to the rate of change:



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Good luck on your exam!

You can use the rest of the time to practice



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