

CARE

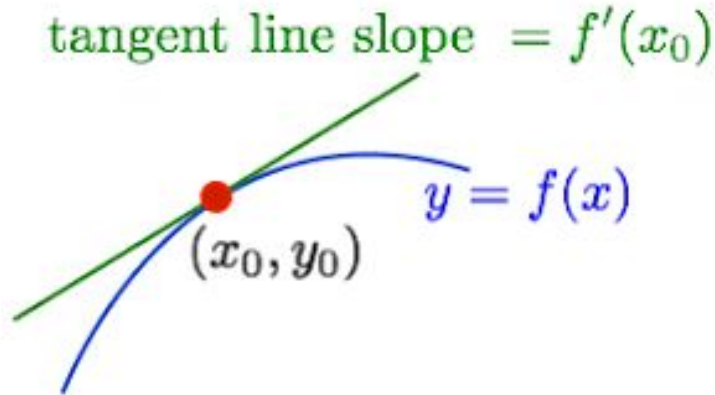
MATH 221 MIDTERM 1 REVIEW



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Slopes of Tangent Lines

AKA: The derivative at a certain point along a curve (function)



Point Slope Form:

Variables
Describe a specific point

$$y - y_1 = m(x - x_1)$$

y coordinate
the y coordinate of a point on the line

Slope
Describes the slope of the line

x coordinate
the x coordinate of a point on the line

Limits

$$\lim_{x \rightarrow a} \overbrace{f(x)}^{\text{function}} = \underbrace{L}_{\text{"What is the y-value getting closer to?"}}$$

“As you approach a along the x -axis”

Limit and Continuity

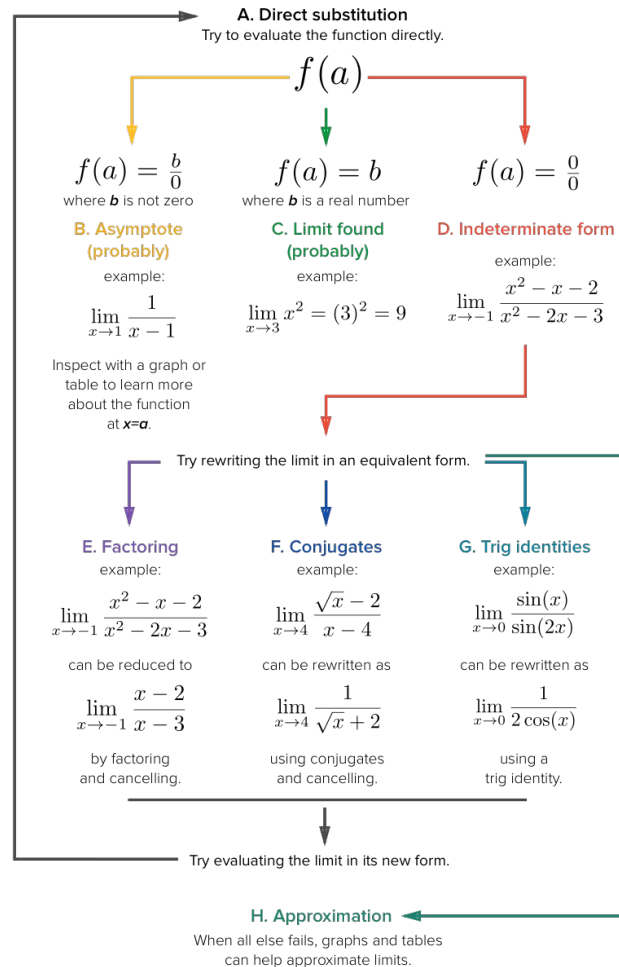
Existence of a Limit

The limit of a function $f(x)$ exists if and only if the one sided limits of the function are equal.

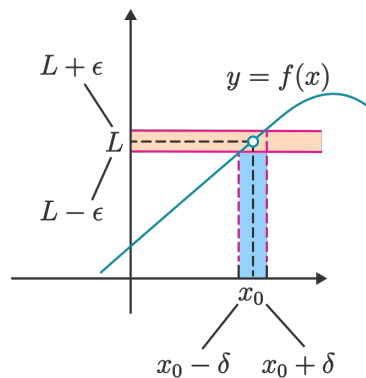
$$\lim_{x \rightarrow c} f(x) = L \quad \text{if and only if}$$

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$$

$$\text{Calculating } \lim_{x \rightarrow a} f(x)$$



Delta-Epsilon Definition



Definition	Translation
1. For every $\epsilon > 0$,	1. For every positive distance ϵ from L ,
2. there exists a $\delta > 0$,	2. There is a positive distance δ from a ,
3. such that	3. such that
4. if $0 < x - a < \delta$, then $ f(x) - L < \epsilon$.	4. If x is closer than δ to a and $x \neq a$, then $f(x)$ is closer than ϵ to L .

Let $f(x)$ be a function defined on the interval that contains $x = a$.

Then $\lim_{x \rightarrow a} f(x) = L$ if for every number $\epsilon > 0$ there exists some real number $\delta > 0$ so that if

$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon$$

Limit Laws

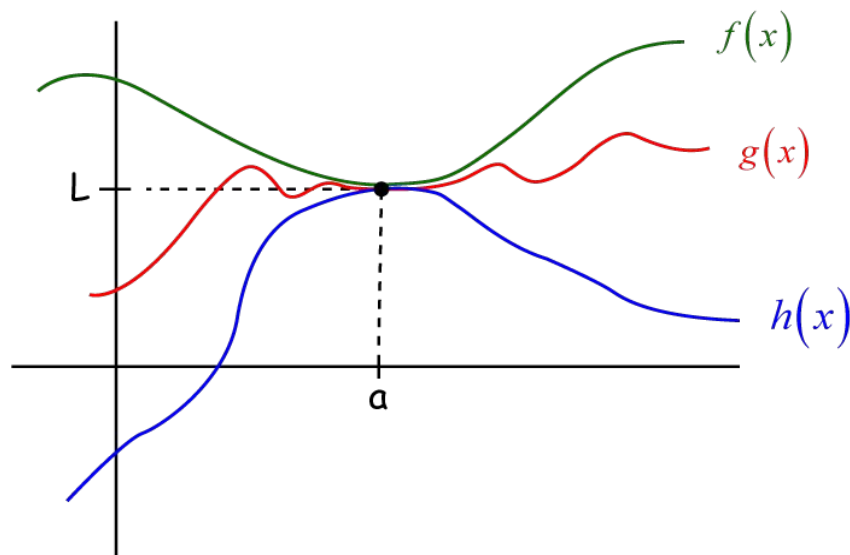
properties of limits

Let a, k, A , and B represent real numbers, and f and g be functions, such that $\lim_{x \rightarrow a} f(x) = A$ and $\lim_{x \rightarrow a} g(x) = B$. For limits that exist and are finite, the properties of limits are summarized in [Table](#)

Constant, k	$\lim_{x \rightarrow a} k = k$
Constant times a function	$\lim_{x \rightarrow a} [k \cdot f(x)] = k \lim_{x \rightarrow a} f(x) = kA$
Sum of functions	$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = A + B$
Difference of functions	$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = A - B$
Product of functions	$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = A \cdot B$
Quotient of functions	$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{A}{B}, B \neq 0$
Function raised to an exponent	$\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$, where n is any real number
n th root of a function , where n is a positive integer	$\lim_{x \rightarrow a} f(x) \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} [f(x)]} = \sqrt[n]{A}$
Polynomial function	$\lim_{x \rightarrow a} p(x) = p(a)$

Squeeze Theory

If $h(x) \leq g(x) \leq f(x)$ when x is near a , except possibly at a ,
and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then $\lim_{x \rightarrow a} g(x) = L$.



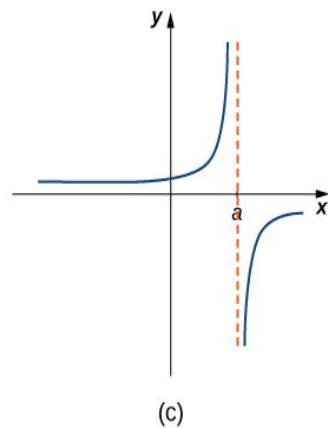
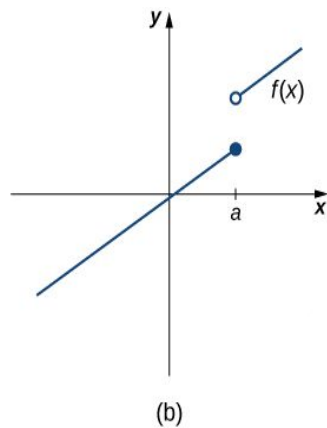
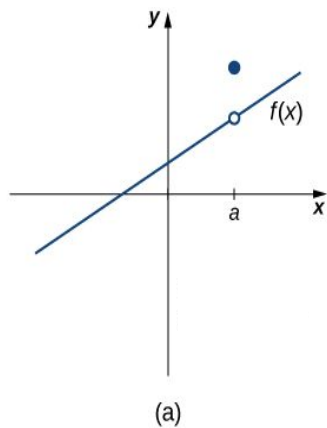
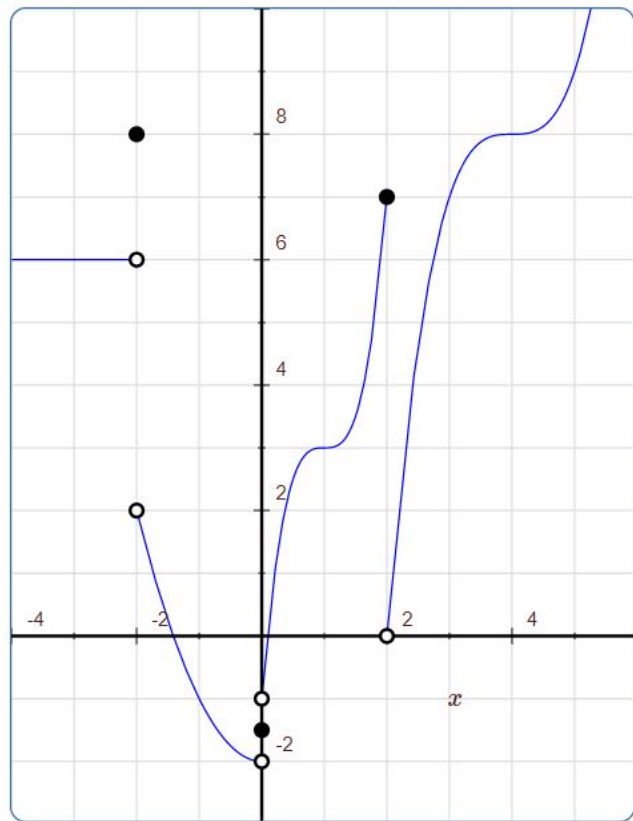
Criteria for Continuity

Continuity

A function is continuous at c when the following three conditions are met:

1. $f(c)$ is defined
2. $\lim_{x \rightarrow c} f(x)$ exists
3. $\lim_{x \rightarrow c} f(x) = f(c)$

Types of Discontinuities



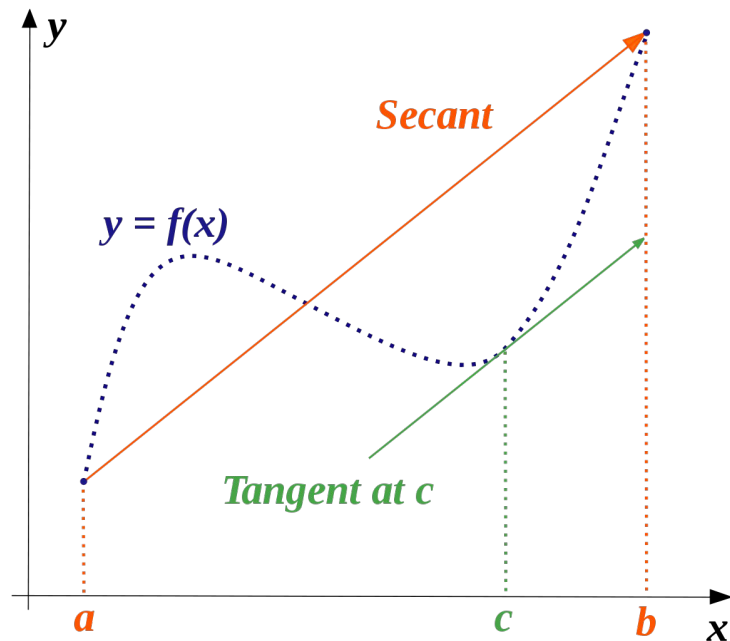
Limit Definition of a Derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

IVT

10 The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



DIFFERENTIABILITY

Definition of Differentiability

25.1 DEFINITION Let f be a real-valued function defined on an interval I containing the point c . (We allow the possibility that c is an endpoint of I .) We say that f is **differentiable** at c (or has a **derivative** at c) if the limit

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

exists and is finite. We denote the derivative of f at c by $f'(c)$ so that

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

whenever the limit exists and is finite. If the function f is differentiable at each point of the set $S \subseteq I$, then f is said to be differentiable on S , and the function $f' : S \rightarrow \mathbb{R}$ is called the derivative of f on S .

Differentiation

Sum rule	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
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Difference rule	$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
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Constant multiple rule	$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x)$
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Constant rule	$\frac{d}{dx}k = 0$
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Differentiation Rules

$$\frac{d}{dx} c = 0$$

Constant Rule

$$\frac{d}{dx} x^n = nx^{n-1}$$

Power Rule

$$\frac{d}{dx} \sin(x) = \cos(x)$$

Trigonometric Rules

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

$$\frac{d}{dx} b^x = b^x \ln(b)$$

Exponential Rule

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

Logarithmic Rule

Product and Quotient Rule

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Trig Derivatives

Trig Derivatives

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \tan x \sec x$$

$$\frac{d}{dx} \csc x = -\cot x \csc x$$

Commit these to memory!!

They will show up a lot
throughout the calculus sequence

Derivatives of Inverse Trig Functions

$$\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$$

$$\frac{d}{dx} \operatorname{arccot} u = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx} \operatorname{arcsec} u = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx} \operatorname{arccsc} u = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Chain Rule

$$\frac{d}{dx} \left[(f(x))^n \right] = n(f(x))^{n-1} \cdot f'(x)$$

$$\frac{d}{dx} [f(g(x))] = f'(g(x))g'(x)$$

Implicit Differentiation

Find $\frac{dy}{dx}$ given:

$$x^3 + y^3 = 6xy$$

Treat y as a function of x , so every y -term requires the chain rule.

$$\frac{d}{dx} [x^3 + y^3] = \frac{d}{dx} [6xy]$$

Left side, term by term:

$$\frac{d}{dx} [x^3] = 3x^2$$

$$\frac{d}{dx} [y^3] = 3y^2 \frac{dy}{dx} \quad (\text{chain rule})$$

Right side requires the product rule, since $6xy$ is a product of two functions of x :

$$\frac{d}{dx} [6xy] = 6 \left(y + x \frac{dy}{dx} \right)$$

Step 2 — Assemble the differentiated equation

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$$

Step 3 — Collect all $\frac{dy}{dx}$ terms on one side

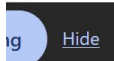
$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

Step 4 — Factor out $\frac{dy}{dx}$

$$\frac{dy}{dx} (3y^2 - 6x) = 6y - 3x^2$$

Step 5 — Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$



Solution

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

Logarithmic Differentiation

General Workflow

1. Take the natural log of both sides
2. Take the derivative of both sides in terms of x
3. Get dy/dx onto one side
4. Plug in the original equation for any y 's in the equation

$y = \sin x^{2x}$	
$\ln y = \ln(\sin x)^{2x}$	Take the natural log of both sides of the equation
$\ln y = 2x \cdot \ln(\sin x)$	Apply logarithmic property $\log_a M^N = N \log_a M$
$\begin{aligned}\frac{d}{dx}[\ln y] &= \frac{d}{dx}[2x \cdot \ln \sin x] \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 2x \cdot \left(\frac{\cos x}{\sin x}\right) + \ln(\sin x) \cdot (2) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 2x \cot x + 2 \ln(\sin x)\end{aligned}$	Differentiate implicitly
$\frac{dy}{dx} = y(2x \cot x + 2 \ln(\sin x))$	Solve for $\frac{dy}{dx}$
$\frac{dy}{dx} = \sin x^{2x} (2x \cot x + 2 \ln(\sin x))$	Substitute $y = \sin x^{2x}$ into final answer

Pre-worksheet

Practice

Example 3 Find the derivative of the following function using the definition of the derivative.

$$R(z) = \sqrt{5z - 8}$$

[Show Solution ▶](#)