

Q4: Consider the statistical mechanics of a quantum harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2 + \hat{z}^2).$$

- a) In this part of the question label the states by quantum numbers n_x , n_y , n_z , that are related to motion in the x , y , z directions. Use this labelling to write down a formula for the energy eigenvalues E_{n_x, n_y, n_z} , and find an expression for the degeneracy of each energy level (*i.e.* the number of states with a given allowed energy).
- b) Compute the partition function for the quantum oscillator, and hence the average energy and entropy of the system at temperature T .
- c) As this system is rotationally invariant, we can uniquely label the energy eigenstates by the total angular momentum ℓ , the eigenvalue m_ℓ of the z component of angular momentum, and a principal quantum number k that takes the values $k = 0, 1, 2, 3, \dots$. In terms of these quantities the energy eigenvalues are

$$E_{k, \ell, m_\ell} = \hbar\omega(2k + \ell + 3/2).$$

(You do *not* have to derive this expression!) Recompute the partition function using this energy formula. Does it equal your partition function from part (b)?

- d) For a system at temperature T , what is the average value of the orbital angular momentum ℓ ? Show that for large T the average value of ℓ rises linearly with T .

Possibly useful formulæ:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \quad \sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n \quad \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{1}{2}(n+1)(n+2)x^n$$