Q1: A globular star cluster has a spherically symmetric mass distribution

$$\rho(r) = \frac{3M_0}{4\pi r_s^3}\left(1+\frac{r^2}{r_s^2}\right)^{-5/2}.$$

a) Write down the Poisson equation which determines the corresponding gravitational potential $\phi(r)$. Assume that $\phi(r) = A(r^2+r_s^2)^{-1/2}$ for some coefficient A and find an expression for A in terms of Newton's constant G and the parameters in $\rho(r)$.

b) Write down, or derive from a Lagrangian, the equations of motion for a particle of mass m moving in a spherically symmetric potential $V(r)$. Show that they lead to the conservation of angular momentum $L \equiv mr^2\dot{\theta}$. Assuming that the particle has angular momentum L, obtain an equation that does not contain θ for the time evolution of the radial coordinate.

c) Show that the equation you found in part (b) can be written as

$$\frac{d^2r}{dt^2} = -\frac{d}{dr}U_{\text{eff}}(r)$$

and find an expression for $U_{\text{eff}}(r)$ in terms of $V(r)$ and L.

d) Suppose that

$$V(r) = -\frac{GMm}{\sqrt{r^2+r_s^2}},$$

(which may or may not be the potential you found in part (a)) and assuming that the particle is a star in a circular orbit of radius r_0, find its angular momentum L.

e) An impact in the radial direction slightly perturbs the motion while leaving L unchanged. As a result the distance $r(t)$ has small amplitude oscillations. What is their angular frequency ω?