Q3: An electron is immersed in a uniform magnetic field. The quantum Hamiltonian can be taken to be

$$\hat{H} = -\gamma \mathbf{S} \cdot \mathbf{B},$$

where $\gamma = -e/m_{\text{electron}}$ is the electron's gyromagnetic ratio, and $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$ with $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Consider first the case where the magnetic field is in the z direction. Suppose that at time $t = 0$ the electron is in a state with its spin component along the x-axis being $\hbar/2$. Determine the expectation values $\langle\psi|S_x|\psi\rangle$, $\langle\psi|S_y|\psi\rangle$, $\langle\psi|S_z|\psi\rangle$ of the x, y, and z components of the electron's spin operators at later time t.

b) Now allow the magnetic field $\mathbf{B}$ to point in the direction specified by the unit vector $\hat{\mathbf{n}}$. Write down, in terms of γ, $\mathbf{B}$, and $\hbar$, the matrix representing the unitary time-evolution operator $U(t)$ that takes an initial state $|\psi(0)\rangle$ to $|\psi(t)\rangle$. Expand it out as

$$U(t) = \alpha(t)\mathbb{I} + \beta(t)\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}, \quad \text{where} \quad \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Find explicit expressions for $\alpha(t)$, $\beta(t)$.

c) A key component of quantum circuit technology is the *Hadamard gate*, which is an operator $\hat{O}$ that acts on a qubit (*i.e.* a two-level quantum system), performing the following transformation of its basis states (traditionally called $|0\rangle$ and $|1\rangle$)

$$|0\rangle \to \hat{O}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |1\rangle \to \hat{O}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Identifying $|0\rangle$ with the spin-1/2 fully polarized along the z-axis, and $|1\rangle$ with its opposite polarization, prove that the Hadamard transform may be implemented by applying to the particle a constant magnetic field

$$\mathbf{B} = \left(\frac{1}{\sqrt{2}}\hat{\mathbf{x}} + \frac{1}{\sqrt{2}}\hat{\mathbf{z}}\right)$$

for a certain time interval τ, and calculate τ.