Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

MATH 257 – Linear Algebra with Computational Applications

Extras Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: May 6, 5:30 - 7:00 PM Aman, Maheen, Serge

Session 2: May 7. 3:30 - 5:00 PM Rohan, Aidan, Siddh

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/955
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. For any $\boldsymbol{A_{nxn}}$ matrix, there exists an invertible matrix P and diagonal matrix D, such that $A = PDP^{-1}$ for all square matrices.

   $\boxed{False}$, A must have a valid eigenbasis and be square. A counterexample would be

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

2. True/False : Another way of representing matrix diagonalization is $L_{BB} = I_{BE}L_{EE}I_{EB}$, where the basis B represents the eigenbasis of a matrix A in the standard basis

   Let A be a collection of vectors $[\boldsymbol{a_1}, \boldsymbol{a_2}, .., \boldsymbol{a_k}] \in \mathbb{R}^n$. A is linearly independent iff Nul(A) $\neq 0$.

   $\boxed{False}$, the representation is actually $L_{EE} = I_{EB}L_{BB}I_{BE}$, because L_{EE} represents the transformation in the standard basis, which is given by A. L_{BB} is the transformation in the eigenbasis.

   True/False : Matrix A that has $\lambda_1 = \lambda_2 = 2$ and linearly independent eigenvectors V_1 and V_2 is diagonalizable.

   $\boxed{True}$, a matrix with an eigenvalue of algebraic multiplicity > 1 is still diagonalizable if the eigenvectors are independent (the geometric multiplicity is equal to the algebraic multiplicty)

3. If A is a 3×3 matrix such that: $A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\3\\0\end{bmatrix}$, $A\begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}-\frac{2}{\sqrt{2}}\\0\\\frac{2}{\sqrt{2}}\end{bmatrix}$, $A\begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$ What is A^2?

Observe that these are eigenvector equations, and the eigenvectors form an eigenbasis:

$$P = \begin{bmatrix}0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\1 & 0 & 0\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} \quad A = PDP^{-1} \text{ so, } A^2 = PD^2P^{-1} \quad D = \begin{bmatrix}3 & 0 & 0\\0 & 2 & 0\\0 & 0 & -1\end{bmatrix}$$

$$A^2 = \begin{bmatrix}0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\1 & 0 & 0\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}9 & 0 & 0\\0 & 4 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\1 & 0 & 0\\0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}^{-1}$$

4. What is the limit of $A^k v$ for all $v \in \mathbb{R}^n$ as $k \to \infty$ if $|\lambda_i| < 1$, where λ_i is an eigenvalue of matrix A?

$\lim_{k\to\infty} A^k v = 0$ because $A^k = PD^kP^{-1}$, and raising $|\lambda| < 1$ to infinitely large power results in a 0 matrix

5. SVD given Diagonalization. **it is not necessary to calculate matrix inverses**

a) Matrix A is diagonalized with the following matrices: $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, D = \begin{bmatrix} 17 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Find the SVD matrices $U, \Sigma,$ and V so that $A = U\Sigma V^\intercal$

Notice that P is an orthogonal matrix, so $A = PDP^\intercal$. This already closely mirrors the SVD format $A = U\Sigma V^\intercal$. However, the singular values in the Σ matrix must be positive, and A has a negative eigenvalue. To solve this problem, we can flip the sign of the eigenvector corresponding to $\lambda = -3$ in either the U or V matrices (but not both!)

So there are 2 possible solutions: $U = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 17 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{bmatrix}, V = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Or: $U = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 17 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{bmatrix}, V = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) Matrix A is diagonalized with the following matrices: $P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -16 \end{bmatrix}$

Find the Compact SVD matrices $U_C, \Sigma_C,$ and V_C so that $A = U\Sigma V^\intercal$

As with the previous problem, the P matrices are orthogonal, but one eigenvalue is 0. Thus A is a rank-2 matrix, and the compact SVD can drop the second column corresponding to $\lambda = 0$. As with the last problem, we swap signs of one eigenvector corresponding to a negative eigenvalue

Thus, $U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma = \begin{bmatrix} 8 & 0 \\ 0 & 16 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$

Or: $U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 8 & 0 \\ 0 & 16 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

6. A is a 3×4 matrix with singular values σ_1 and σ_2. The column space of A is given $col(A) = span \left\{ \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$, and the null space is given $null(A) = span \left\{ \begin{bmatrix} -4 \\ 9 \\ -2 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -1 \\ 4 \end{bmatrix} \right\}$

Produce the compact SVD matrices of A

Given the spanning set of the column space, the matrix is rank 2. Thus the compact SVD matrices will have the following shapes:
$U_C : 3 \times 2$
$\Sigma_C : 2 \times 2$
$V_C : 4 \times 2$

Σ_C is simply a diagonal matrix containing the singular values, thus $\boxed{\Sigma_C = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}$

U contains an ONB for the column space in the first two columns, and an ONB for the left-null space in the last column. However, U_C does not contain information about the left-null space, so the given basis for $col(A)$ can be normalized. This gives $\boxed{U_C = \begin{bmatrix} \frac{9}{\sqrt{115}} & \frac{-2}{\sqrt{14}} \\ \frac{5}{\sqrt{115}} & \frac{3}{\sqrt{14}} \\ \frac{3}{\sqrt{115}} & \frac{1}{\sqrt{14}} \end{bmatrix}}$

Similarly to U_C, V_C contains an ONB of the row space, whereas V contains a basis of the row space and the null space. Because the null-space is given, the basis of the row space can be calculated as its orthogonal complement.

Through a method of your choosing you may find: $col(A) = span \left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$

Normalizing these vectors give the V_C matrix: $\boxed{\text{V\_C} = \begin{bmatrix} \frac{4}{\sqrt{30}} & \frac{-1}{\sqrt{20}} \\ \frac{3}{\sqrt{30}} & \frac{0}{\sqrt{20}} \\ \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{20}} \\ \frac{1}{\sqrt{30}} & \frac{0}{\sqrt{20}} \end{bmatrix}}$