



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 241 – Calculus III

CBTF Midterm 5 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: May 3, 3-4:30 pm, 433 Grainger, Meredith and Rodrigo

Can't make it to a session? Here's our schedule by course:

<https://care.grainger.illinois.edu/tutoring/schedule-by-subject>

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: <https://queue.illinois.edu/q/queue/845>
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Consider the following vector fields $\vec{F}(x, y, z)$. Are they conservative? If so, find a function $f(x, y, z)$ so that $\nabla f = \vec{F}$. If not, justify your response.

(a) $\vec{F}(x, y, z) = \langle yz, xz, xy + 2z \rangle$

(b) $\vec{F}(x, y, z) = \langle y + e^x, x - \cos y, 4 + z \rangle$

(c) $\vec{F}(x, y, z) = \langle y, z^2, x \rangle$

Conservative vector field test: a vector field \vec{F} is conservative if the curl is the zero vector.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right\rangle = \vec{0}$$

(a)

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy + 2z \end{vmatrix} = \langle x - x, y - y, z - z \rangle = \vec{0}$$

The vector field is conservative, therefore, a potential function exists. To find it, we must find the necessary terms from each component (We neglect the constant for now, we'll add it back later).

$$\int F_x \, dx = \int yz \, dx = xyz$$

$$\int F_y \, dy = \int xz \, dy = xyz$$

$$\int F_z \, dz = \int xy + 2z \, dz = xyz + z^2$$

We see that the necessary terms are xyz and z^2 , therefore

The field is conservative and has potential function $f(x, y, z) = xyz + z^2 + C$

(b)

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + e^x & x - \cos y & 4 + z \end{vmatrix} = \langle 0 - 0, 0 - 0, 1 - 1 \rangle = \vec{0}$$

The vector field is conservative, therefore, a potential function exists. To find it, we must integrate each component (We neglect the constant for now, we'll add it back later).

$$\int F_x \, dx = \int y + e^x \, dx = xy + e^x$$

$$\int F_y \, dy = \int x - \cos y \, dy = xy - \sin y$$

$$\int F_z \, dz = \int 4 + z \, dz = 4z + \frac{1}{2}z^2$$

We see that the necessary terms are xy , e^x , $-\sin y$, $4z$, and $\frac{1}{2}z^2$, therefore:

The field is conservative and has potential function $f(x, y, z) = xy + e^x - \sin y + 4z + \frac{1}{2}z^2 + C$

(c)

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z^2 & x \end{vmatrix} = \langle -2z, -1, -1 \rangle$$

This vector field is not conservative. Therefore, a potential function does not exist.

2. The vector field $\vec{F} = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$ is conservative. Find a potential function f for \vec{F} (a function with $\nabla f = \vec{F}$)

$$\vec{F} = \langle f_x, f_y \rangle = \langle 2xy + 2x + y^2, 2xy + 2y + x^2 \rangle$$

$$\int f_x = yx^2 + x^2 + xy^2 + C(y)$$

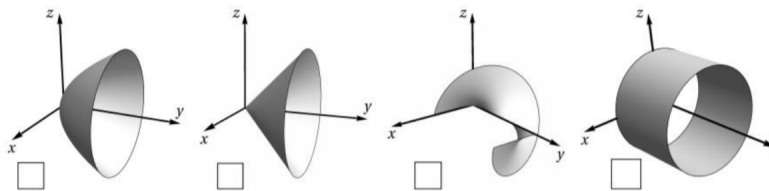
$$\int f_y = xy^2 + y^2 + yx^2 + C(x)$$

Looking at all the terms and comparing with \vec{F} , we know that $C(y) = y^2$ and $C(x) = x^2$, therefore the potential function is:

$$f(x, y, z) = yx^2 + x^2 + xy^2 + y^2$$

3. Let S be the surface parameterized by $\vec{r}(u, v) = \langle v \cos(u), v, v \sin(u) \rangle$ for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$

(a) Mark the picture of S below



(b) Evaluate the surface integral $\iint_S y \, dS$

- (a) There must be circular cross section on the xz plane that have varying radius that depend on v therefore it is one of the first two options. Furthermore, the parameterization satisfies $x^2 + z^2 = y^2$ which is the equation of the cone (second image).

The answer is then the second option

- (b) To do this surface integral, you have to compute $dS = |\vec{r}_u \times \vec{r}_v|$ and replace y with the vector component, which is v here.

$$\iint_S y \, dS = \int_0^{2\pi} \int_0^1 v |\vec{r}_u \times \vec{r}_v| \, dv du$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v \sin u & 0 & v \cos u \\ \cos u & 1 & \sin u \end{vmatrix} = \langle -v \cos u, v, -v \sin u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^2 \cos^2 u + v^2 + v^2 \sin^2 u} = \sqrt{v^2(\cos^2 u + \sin^2 u + 1)} = \sqrt{2}v$$

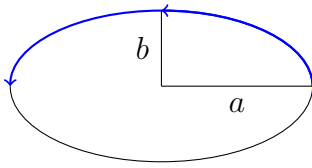
Then integrate this with the correct bounds for u and v .

$$\int_0^{2\pi} \int_0^1 v |\vec{r}_u \times \vec{r}_v| \, dv du = \int_0^{2\pi} \int_0^1 \sqrt{2}v^2 \, dv du = \int_0^{2\pi} \left. \frac{\sqrt{2}}{3}v^3 \right|_{v=0}^{v=1} du$$

$$\int_0^{2\pi} \frac{\sqrt{2}}{3} du = \frac{2\sqrt{2}}{3}\pi$$

$$\iint_S y \, dS = \boxed{\frac{2\sqrt{2}}{3}\pi}$$

4. A particle moves along the upper part of an ellipse in the xy -plane that has its center at the origin with semi-major and semi-minor axes $a = 4$ and $b = 3$, respectively. Starting at $(a, 0, 0)$ and ending at $(-a, 0, 0)$ and subject to the following force field, what is the total work done?



$$\vec{F} = (3x - 4y + 2z)\hat{i} + (4x + 2y - 3z^2)\hat{j} + (2xz - 4y^2 + z^3)\hat{k}$$

- (1) Find the parameterization of the ellipse

$$x = 4 \cos t, \quad y = 3 \sin t, \quad z = 0$$

$$dx = -4 \sin t \, dt, \quad dy = 3 \cos t \, dt, \quad dz = 0$$

Recall that $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

- (2) Take the dot product in the line integral for work. (The z component is zero so it can be ignored here).

$$\begin{aligned} \oint \vec{F} \cdot d\vec{r} &= \int [(3x - 4y)\hat{i} + (4x + 2y)\hat{j}] \cdot (dx\hat{i} + dy\hat{j}) \\ &= \oint (3x - 4y)dx + (4x + 2y)dy \end{aligned}$$

- (3) Substitute in the parameterization found in (1)

$$\oint (12 \cos(t) - 12 \sin(t))(-4 \sin(t))dt + (16 \cos(t) + 6 \sin(t))(3 \cos(t))dt$$

- (4) Determine the times that the particle is at its starting and ending position, which in this case is $0 < t < \pi$. And solve the integral

$$\int_{t=0}^{t=\pi} [(12 \cos t - 12 \sin t)(-4 \sin t) + (16 \cos t + 6 \sin t)(3 \cos t)] dt$$

$$\boxed{48\pi}$$

5. Find the work done by the force field below in moving an object from (1,1) to (2,4) (HINT: Check if the vector field is conservative).

$$\vec{F}(x, y) = \langle 6y^{\frac{3}{2}}, 9x\sqrt{y} \rangle$$

First we need to check if this vector field is conservative

$$\frac{\partial P}{\partial y} = 9\sqrt{y} \text{ and } \frac{\partial Q}{\partial x} = 9\sqrt{y}$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ we can say the vector field is conservative

Now we can find a function $f(x, y)$ such that $\nabla f = \vec{F}$

$$\int F_x dx = \int 6y^{\frac{3}{2}} dx = 6xy^{\frac{3}{2}}$$

$$\int F_y dy = \int 9x\sqrt{y} dy = 6xy^{\frac{3}{2}}$$

Thus our potential function is

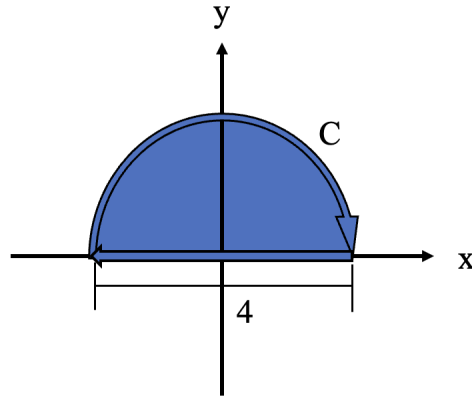
$$f(x, y) = 6xy^{\frac{3}{2}} + C$$

Now that we have a potential function we can use the Fundamental Theorem of Line Integrals to compute the work done in moving from (1,1) to (2,4)

$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f d\vec{r} = f(x_2, y_2) - f(x_1, y_1) = \\ &f(2, 4) - f(1, 1) = (96 + C) - (6 + C) \end{aligned}$$

$$\boxed{W = 90 \text{ (units)}}$$

6. Evaluate $\int_C F \cdot dr$ where $F(x, y) = \langle 3y^2 - \cos y, x \sin y \rangle$ and C is a clockwise path shown below.



Green's Theorem states that $\int_C F \cdot dr = \oint_{-C} Pdx + Qdy = - \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$
 (It is $-C$ because the circle is oriented in a clockwise direction.)

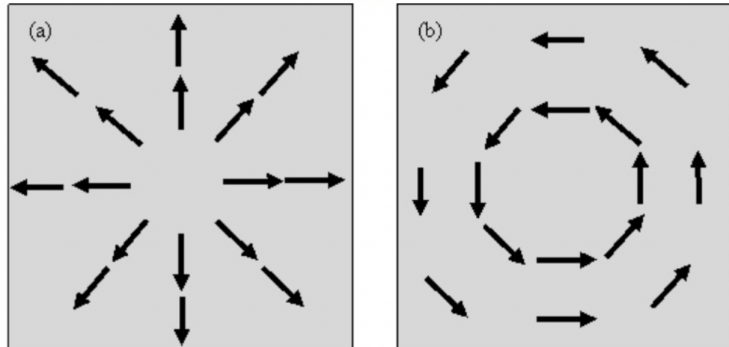
$$Q = x \sin y, P = 3y^2 - \cos y$$

$$- \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = - \iint_D \sin y - (6y + \sin y) dA = \iint_D 6y dA$$

Use polar coordinates:

$$\int_0^\pi \int_0^2 (6r \sin \theta) r dr d\theta = \int_0^\pi 16 \sin \theta d\theta = 16[\cos(0) - \cos(\pi)] = \boxed{32}$$

7. The graph below shows two vector fields. Answer the following questions for each of them.
- (1) Is it a conservative vector field?
 - (2) Does it have a positive, negative, or zero curl?
 - (3) Does it have a positive, negative, or zero divergence?



For vector field (a): (1) It is a conservative vector field because the line integral along any closed path is 0. (2) It has a zero curl because the vectors are not rotating. (3) It has a positive divergence because the vectors have the tendency to diverge out from a point.

For vector field (b): (1) It is not a conservative vector field because the line integral along the closed path is nonzero. (2) It has a positive curl as vectors are rotating in the counterclockwise direction. (3) It has a zero divergence because the vectors are not diverging from a single point.

8. Evaluate the flux of the vector field $F(x, y, z) = \langle x, y, xy \rangle$ where the surface S is part of the paraboloid $z = 4 - x^2 - y^2$ that lies within $0 \leq x \leq 1, 0 \leq y \leq 1$, and is oriented upwards.

The surface can be represented in the vector form:

$$\begin{aligned}\vec{r}(u, v) &= \langle u, v, 4 - u^2 - v^2 \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1 \\ \vec{r}_u &= \langle 1, 0, -2u \rangle \\ \vec{r}_v &= \langle 0, 1, -2v \rangle \\ \vec{r}_u \times \vec{r}_v &= \langle 2u, 2v, 1 \rangle \rightarrow \text{orients upwards!} \\ \vec{F} &= \langle u, v, uv \rangle\end{aligned}$$

The flux can then be evaluated as:

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA \\ &= \int_0^1 \int_0^1 \langle u, v, uv \rangle \cdot \langle 2u, 2v, 1 \rangle dx dy = \int_0^1 \int_0^1 2u^2 + 2v^2 + uv du dv \\ &= \int_0^1 \frac{2}{3} + 2v^2 + \frac{v}{2} dv = \boxed{\frac{19}{12}}\end{aligned}$$

9. Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f(x, y) = ye^{x^2-1} + 4x\sqrt{y}$ and C is given by $\vec{r}(t) = \langle 1 - t, 2t^2 - 2t \rangle$ with $0 \leq t \leq 2$.

Use the fundamental theorem of line integral.

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= f[\vec{r}(2)] - f[\vec{r}(0)] \\ \vec{r}(0) &= \langle 1, 0 \rangle, \text{ and } \vec{r}(2) = \langle -1, 4 \rangle \\ f[\vec{r}(2)] - f[\vec{r}(0)] &= f(-1, 4) - f(1, 0) = -4 - 0 = \boxed{-4}\end{aligned}$$

10. Compute the following surface integral $\iint_S xy \, dS$ where S is the region of the plane $x + y + z = 1$ in the first octant.

We are solving

$$\iint_S xy \, dS = \iint_S xy |\mathbf{r}_u \times \mathbf{r}_v| \, dudv$$

First, we parameterize the plane as $\mathbf{r}(u, v) = \langle u, v, 1 - u - v \rangle$ with $0 \leq u \leq 1 - v$ and $0 \leq v \leq 1$ and plug it into the vector field to get $xy = uv$. We know must compute the cross product term

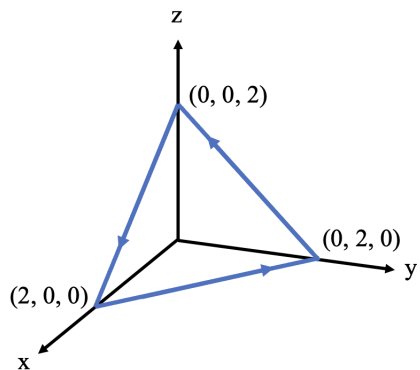
$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

The magnitude of this cross product is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

The bounds of u and v are determined by x and y because $u = x$ and $x = y$. Take the triangle on the xy -plane when $z=0$, and it can be seen that $0 \leq x \leq 1 - y$ and $0 \leq y \leq 1$. Therefore, $0 \leq u \leq 1 - v$ and $0 \leq v \leq 1$. Now we compute

$$\begin{aligned} \iint_S xy \, dS &= \iint_S uv |\mathbf{r}_u \times \mathbf{r}_v| \, dudv \\ &= \sqrt{3} \int_0^1 \int_0^{1-v} uv \, dudv \\ &= \sqrt{3} \int_0^1 \frac{1}{2}(1-v)^2 v \, dv \\ &= \frac{\sqrt{3}}{2} \int_0^1 v - 2v^2 + v^3 \, dv \\ &= \frac{\sqrt{3}}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\ &= \boxed{\frac{\sqrt{3}}{24}} \end{aligned}$$

11. Use Stokes' theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle y^2, x, z \rangle$ across the curve C shown in the figure below.



Stokes' theorem states that $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot \vec{n} dS = \iint_D \text{curl} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA$

$$\text{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z \end{vmatrix} = \langle 0, 0, 1 - 2y \rangle$$

Parameterize the surface using $x = u$ and $y = v$:

$$\vec{r}(u, v) = \langle u, v, 2 - u - v \rangle$$

$$\vec{r}_u = \langle 1, 0, -1 \rangle$$

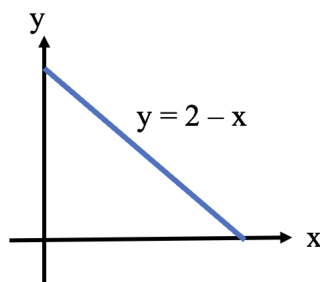
$$\vec{r}_v = \langle 0, 1, -1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 1, 1, 1 \rangle$$

Note that the direction of the cross product matches with the direction of the curve. Then, using the parameterization

$$\text{curl} \vec{F} = \langle 0, 0, 1 - 2v \rangle$$

The bounds of u and v can be determined by looking at the xy -plane shown below.



Looking at the region, $0 \leq y \leq 2 - x$ and $0 \leq x \leq 2$. Therefore, $0 \leq v \leq 2 - u$ and $0 \leq u \leq 2$. Putting everything back together gives us:

$$\iint_D \text{curl} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA = \int_0^2 \int_0^{2-u} \langle 0, 0, 1 - 2v \rangle \cdot \langle 1, 1, 1 \rangle dv du$$

$$\begin{aligned} &= \int_0^2 \int_0^{2-u} 1 - 2v \, dv \, du = \int_0^2 2 - u - (2 - u)^2 \, du = \int_0^2 -2 + 3u - u^2 \, du \\ &= -2u + \frac{3}{2}u^2 - \frac{1}{3}u^3 \Big|_0^2 = \boxed{\frac{-2}{3}} \end{aligned}$$

12. Use Divergence theorem to evaluate the flux of $\vec{F} = \langle x^3 + z^3, x^3 + y^3, y^3 + z^3 \rangle$ across the portion of a sphere with radius of 1 that is centered at origin and above the xy-plane.

Divergence theorem states that $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} dV$

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 3x^2 + 3y^2 + 3z^2$$

Use spherical coordinates: $x^2 + y^2 + z^2 = \rho^2, dV = \rho^2 \sin \phi d\rho d\theta d\phi$

$$\iiint_E \operatorname{div} \vec{F} dV = \iiint_E 3\rho^2 \rho^2 \sin \phi d\rho d\theta d\phi$$

The volume of the sphere is bounded by:

$$0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}$$

$$\begin{aligned} \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 3\rho^4 \sin \phi d\rho d\theta d\phi &= \int_0^{\pi/2} \int_0^{2\pi} \frac{3}{5} \sin \phi d\theta d\phi \\ &= \int_0^{\pi/2} \frac{6\pi}{5} \sin \phi d\phi = \frac{6\pi}{5} (-\cos \frac{\pi}{2} + \cos 0) = \boxed{\frac{6\pi}{5}} \end{aligned}$$