The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 5 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Line Integral Along a Curve with respect to...

- Arc length (orientation does not matter, **integral of C = integral of –C**)

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- x, y (orientation matters, **integral of C = –integral of –C**)

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Line Integral of Vector Fields

- Let F be a continuous vector field defined on a curve C given by a vector function r(t), a ≤ t ≤ b. Line integral of F along C (**Work done**) is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy + R\, dz$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$

Fundamental Theorem of Line Integrals

- Let C be a smooth curve given by the vector function r(t), $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then:

$$\int_C \nabla f \cdot dr = f[r(b)] - f[r(a)]$$

Green's Theorem

- Let C be a **counterclockwise**, **simple closed curve** in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- Green's theorem to calculate the area of a region D bounded by C

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

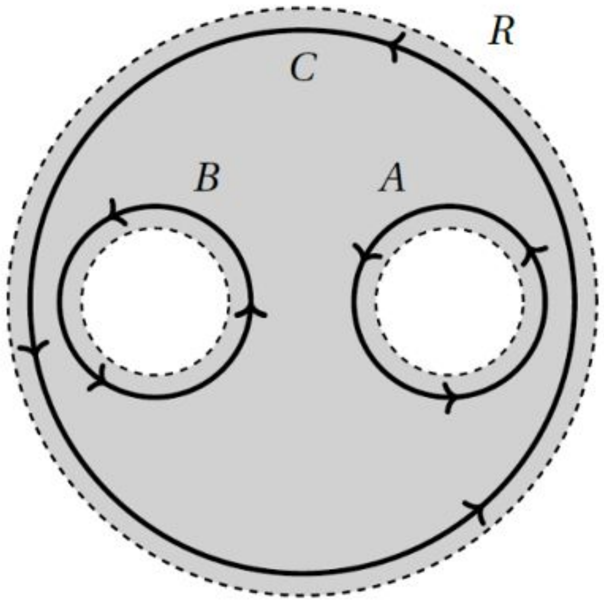
Example Question #2

- Consider the region R shown at the right which contains simple closed curves A, B, and C. Suppose F = <P, Q> is a vector field with continuous partial derivatives on R with the following characteristics:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \qquad \int_A F \cdot dr = 2 \qquad \int_B F \cdot dr = -1$$

(a) Find $\int_C F \cdot dr$

(b) Is this vector field conservative?

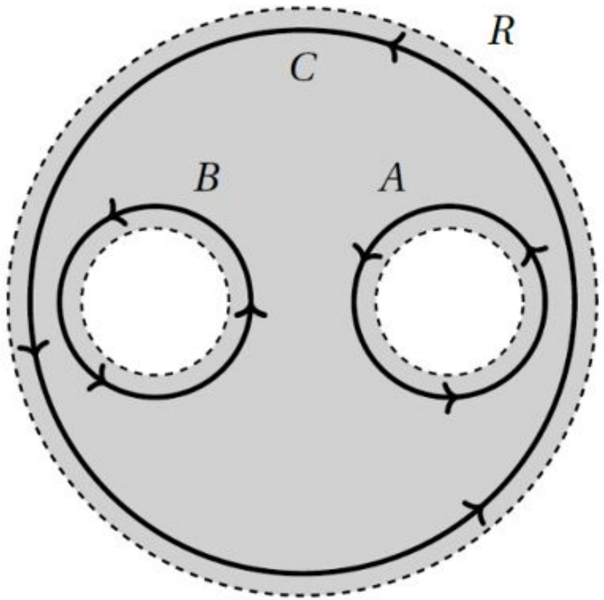


Example Solution #2

(a) Let D be the region enclosed by C. Using Green's theorem:

$$\int_C F \cdot dr - \int_A F \cdot dr - \int_B F \cdot dr = 0$$

$$\int_C F \cdot dr - 2 - (-1) = 0 \qquad \boxed{\int_C F \cdot dr = 1}$$



(b) This vector field is not conservative because it is not a simply-connected region, and the line integral for the closed curve C is not 0.

Curl

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

- Cross product → Curl is a **vector field**
- Describes how vectors **rotate** around a certain point
- Use **right-hand rule** to determine the sign of curl
- Curl of a gradient field = 0
- If F is conservative, curl = 0
- Green's theorem in vector form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA$$

Curl Test for Conservative Vector Field

- If F is a vector field defined on all of R^3 whose component functions have **continuous partial derivatives** and **curl F = 0**, then F is a conservative vector field

Divergence

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

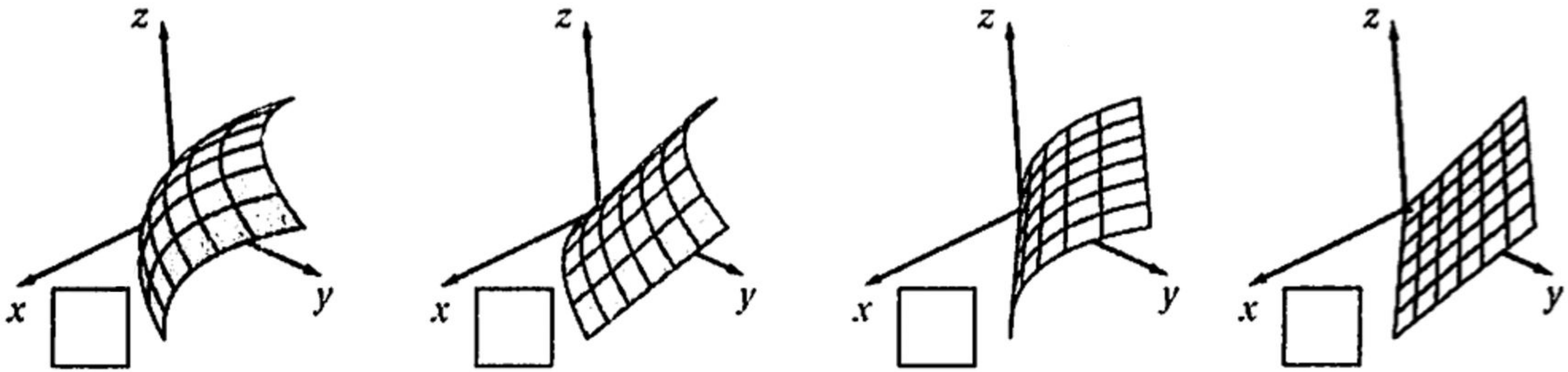
- Dot product → Divergence is a **scalar** field
- Describes how vectors diverge from a single point (or converge to a point)
- **Diverging vectors: positive, Converging vectors: negative**
- Green's theorem in vector form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \text{div}\,\mathbf{F}(x, y)\, dA$$

Example Problem #3

- **Match the surfaces below with the following parametrization:**

$$r(u,v) = < u, u^2 + v^2, v > \text{ defined on } D = \{(u,v) | 0 \leq u \leq 1, 0 \leq v \leq 1\}$$



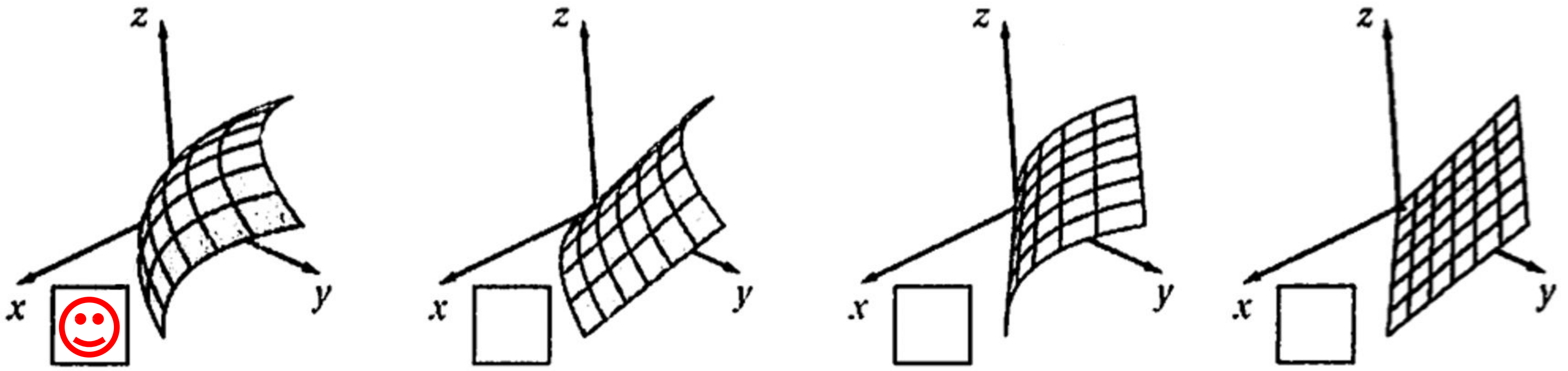
Example Solution #3

$r(u, v) = < u, u^2 + v^2, v >$ defined on $D = \{(u, v) | 0 \leq u \leq 1, 0 \leq v \leq 1\}$

When x is constant → curve on the yz-plane should be a parabola

When y is constant → curve on the xz-plane should be a circle

When z is constant → curve on the xy-plant should be a parabola



Surface Area of a Parametric Surface

- If a parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} \qquad (u, v) \in D$$

, the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

, where r_u and r_v are partial derivatives with respect to u and v.

Surface Integral

- The surface integral of a function f over a parametric surface is:

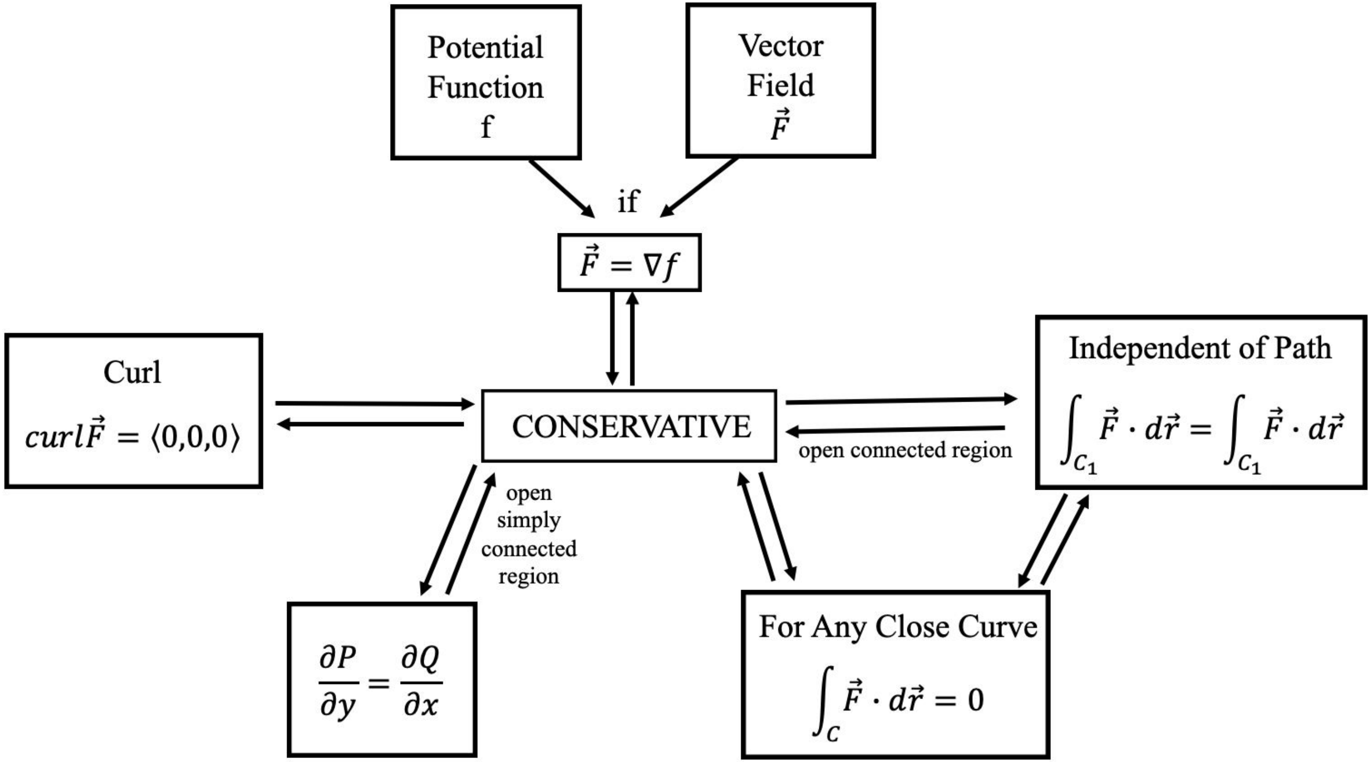
$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Flux

- The flux of a vector field F over a parametric surface is:

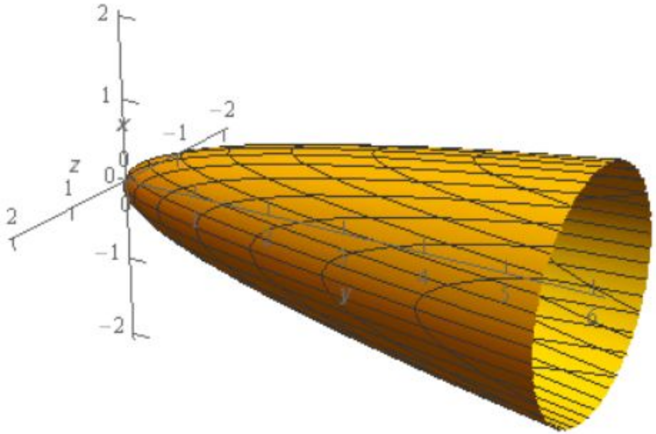
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot (\vec{r_u} \times \vec{r_v})\, dA$$

Conservative Vector Field



Example Problem #6

Evaluate $\iint\limits_S 40y\,dS$ where S is the portion of $y = 3x^2 + 3z^2$ that lies behind $y = 6$.



Example Solution #6

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,g(x,z),z)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + 1 + \left(\frac{\partial g}{\partial z}\right)^2}\,dA$$

In this case D will be the circle/disk we get by setting the two equations equal or,

$$6 = 3x^2 + 3z^2 \qquad \Rightarrow \qquad x^2 + z^2 = 2$$

So, D will be the disk $x^2 + z^2 \leq 2$.

$$\iint_S 40y\,dS = \iint_D 40\left(3x^2 + 3z^2\right)\sqrt{(6x)^2 + 1 + (6z)^2}\,dA$$

Example Solution #6

- Change to polar coordinates for easier computation
-

$$x^2 + z^2 \leq 2.$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq \sqrt{2}$$

$$x^2 + z^2 = r^2$$

$$\iint_D 120\left(x^2+z^2\right)\sqrt{36\left(x^2+z^2\right)+1}\,dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} 120r^3\sqrt{36r^2+1}\,dr\,d\theta$$

Stokes' Theorem

- Let S be a surface that is bounded by a simple, counterclockwise boundary C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS$$

- For a conservative vector field, curl F = 0 → Line integral = 0
- Stokes' Theorem connects the **line integral** of a vector field F along a closed curve C to the **surface integral** of the curl of F over a surface S bounded by C.
- C: Simple, closed curve oriented counterclockwise.
- S: Smooth, oriented surface bounded by CCC.
- n: Unit normal vector to the surface SSS.
- dS: Differential surface element.

Divergence Theorem

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

- Dot product → Divergence is a scalar field
- Describes how vectors diverge from a single point (or converge to a point)
- **Diverging vectors: positive, Converging vectors: negative**
- Green's theorem in vector form:

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Applications of Divergence Theorem

$$\vec{F} = -\frac{GM\vec{r}}{|\vec{r}|^3}$$

If S is <u>any</u> closed surface in $\mathbb{R}^3$ that has M inside it,

$$\iint_S \vec{F}\cdot\vec{n}\,dS = -4\pi GM$$

Gauss' Law

- When given shape is a shell
- Electric flux out of a closed surface depends only on how much electric charge is inside that surface.

If S is a closed surface in $\mathbb{R}^3$ and $\vec{F}$ is the gravitational field

$$\text{Flux}_S(F) = \iint_S \vec{F} \cdot \vec{n}\, dS = -4\pi G M_S$$

where M_S is the total mass enclosed by S.

Gauss' Law

$$\text{Flux of } \vec{E} \text{ through } S = \iint_S \vec{E} \cdot \vec{n}\, dS$$

$$\iint_S \vec{E} \cdot \vec{n}\, dS = \frac{1}{\varepsilon_0} \iiint_V \rho\, dV$$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_V (\nabla \cdot \vec{F})\, dV$$