



MATH 285

Midterm 3 Review

CARE




Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.



Topics

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- I. Systems of Differential Equations
 - I. System notation
 - II. Variation of Parameters
 - III. Eigenvectors/Eigenvalues
 - IV. Diagonalization
 - V. Putzer's Method
 - II. Boundary Value Problems
 - I. Eigenfunction Problems
 - III. Fourier Series



Systems of Ordinary Differential Equations

- Many **physical phenomena** can be described by a coordinated system of differential equations
- For example, **Maxwell's Equations:**

$$\frac{\partial B}{\partial t} = -\nabla \times E$$

$$\mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \nabla \times B$$

- Also, **higher order differential equations** can be broken down into **systems** of ODE's



Creating Systems

- General process:
 - **Redefine a derivative** as a new variable
 - Create **vectors** v , g , and the **matrix** A
 - Write the differential equation in general form:

$$\frac{dv}{dt} = A(t)v + g(t)$$



Existence and Uniqueness

- If the equation is **linear**:

$$\frac{dv}{dt} = A(t)v + g(t)$$

- A **unique solution exists** if $A(t)$ and $g(t)$ are **defined** on the interval



The Fundamental Solution Matrix

- We want to find **n solutions** to the system of differential equations, where **each solution $v_i(t)$ is an n -vector**
- Build the fundamental solution matrix $M(t)$ by **column-stacking each solution $v_i(t)$**

$$v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad M = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$



The Fundamental Solution Matrix

- The **Wronskian** is the **determinant of $M(t)$**
 - If $W(t) \neq 0$, then a solution exists
 - If coefficients are continuous, then $W(t)$ is either identically zero or nonzero
- $M(t)$ depends on how the solutions are chosen
 - **Most convenient choice is $M(t_0) = I$**
 - $M(t_0) = I$ can be calculated as $M(t)M^{-1}(t_0)$



Putting it Together

- The **homogeneous solution** to the following differential equation:

$$\frac{dv}{dt} = A(t)v \quad v(t_0) = v_0$$

is given by:

$$v(t) = M(t)M^{-1}(t_0)v_0$$



Abel's Theorem Extended

- For a system of linear differential equations given by

$$\frac{dv}{dt} = A(t)v$$

$$\frac{dW}{dt} = \text{Tr}(A(t))W$$

- **W** is the **Wronskian** / determinant of the fundamental solution matrix
- The "**trace**" (**Tr**) of a matrix is the **sum of its diagonal components**
- If $A(t)$ is continuous, then the **Wronskian is either always or never 0**

Variation of Parameters

- For a system of differential equations given by:

$$\frac{dv}{dt} = A(t)v + g(t)$$

- The **general solution** is given by:

$$v(t) = \underbrace{M(t) \int_{t_0}^t M^{-1}(s)g(s)ds}_{\text{particular solution}} + \underbrace{M(t)M^{-1}(t_0)v_0}_{\text{characteristic solution}}$$



Calculating Eigenvalues and Eigenvectors

- **Eigenvalues** (λ) and **eigenvectors** (v) are given by:

$$(A - \lambda I)v = 0$$

- Calculate **eigenvalues** with: $\det(A - \lambda I) = 0$
- Then, calculate **eigenvectors** with **first equation**



Matrix Exponentials

- The matrix exponential definition comes from the **power series** definition of an exponent:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{A^2 t^2}{2!} + \dots$$



Constant Coefficient Linear Systems

- If the matrix A is **no longer a function of t** :

$$\frac{dv}{dt} = Av$$

- The **fundamental solution matrix** can be calculated by the **matrix exponential**:

$$M(t) = e^{At}$$

Diagonalization

- If there are n **linearly independent eigenvectors**, then the matrix exponential can be calculated as:

$$e^{At} = UDU^{-1}$$
$$U = [v_1 \quad v_2 \quad \dots], \quad D = \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & \dots \end{bmatrix}$$

Diagram illustrating the diagonalization of the matrix exponential e^{At} . The matrix U is formed by the eigenvectors v_1, v_2, \dots . The matrix D is a diagonal matrix with entries $e^{\lambda_1}, e^{\lambda_2}, \dots$ on the diagonal, representing the exponential of the eigenvalues. Arrows indicate that the columns of U are eigenvectors and the diagonal entries of D are eigenvalues.

Putzer's Method

- Putzer's Method will **always work** for solving the matrix exponential
- Trying to get to:

$$e^{At} = B_0 r_1 + B_1 r_2 + B_2 r_3 \dots$$

- **Number of terms matches degree of matrix**, I.E. 2x2 matrix goes up to the $B_1 r_2$ term



Putzer's Method Contd.

- First, **calculate the B matrices**
- Follow the pattern:

$$B_0 = I$$

$$B_1 = (A - \lambda_1 I)B_0$$

$$B_2 = (A - \lambda_2 I)B_1$$

...

Putzer's Method Contd.

- Second, **calculate the r functions**, then plug everything in
- Follow the pattern:

$$\frac{dr_1}{dt} = \lambda_1 r_1, \quad r_1(0) = 1$$

$$\frac{dr_2}{dt} = \lambda_2 r_2 + r_1, \quad r_2(0) = 0$$

$$\frac{dr_3}{dt} = \lambda_3 r_3 + r_2, \quad r_3(0) = 0$$

...



Putzer's Method Notes and Example

- Putzer's Method **doesn't require knowing the eigenvectors**
- More calculation heavy than diagonalization, but **no matrix multiplication** either
- Example:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Putzer's Method Notes and Example

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_1 = A - 0I = A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$B_2 = (A - 2I)B_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Putzer's Method Notes and Example

$$\frac{dr_1}{dt} = 0r_1 \qquad r_1(0) = 1$$

$$r_1(t) = e^{0t} = 1$$

$$\frac{dr_2}{dt} = 2r_2(t) + 1 \qquad r_2(0) = 0$$

$$r_2(t) = \frac{e^{2t}}{2} - \frac{1}{2}$$

Ordinarily we would have to compute the solution to

$$\frac{dr_3}{dt} = 2r_3 + r_2(t) = 2r_3 + \frac{e^{2t}}{2} - \frac{1}{2} \qquad r_3(0) = 0$$

Putzer's Method Notes and Example

$$e^{tA} = 1 \cdot B_0 + \frac{e^{2t} - 1}{2} B_1 = \begin{pmatrix} \frac{e^{2t}+1}{2} & \frac{e^{2t}-1}{2} & 0 \\ \frac{e^{2t}-1}{2} & \frac{e^{2t}+1}{2} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$$




Boundary Value Problems

- A boundary value problem is an **analog to an initial value problem**
- Instead of specifying just an initial condition, **multiple “boundary” values are specified** to constrain the solution



Eigenvalue Problems

- Eigenvalue problems are boundary value problems with an **unknown parameter λ**
 - We look for cases that have **non-trivial (non-zero) solutions**
 - Generally a good idea to look at the positive case first
 - Final answer is a set of **eigenvalues** and the **eigenfunctions** (solution functions) that go with it
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Eigenvalue Problem Example

$$y'' + \lambda y = 0 \quad \lambda = k^2 \quad y(0) = 0 \quad y'(1) = 0$$

Case 1: $\lambda > 0$ $y = A\cos(kx) + B\sin(kx)$ $\lambda = \left(n + \frac{1}{2}\right)^2 \pi^2$

Case 2: $\lambda = 0$ $y = A + Bx$ $\lambda = 0$

Case 3: $\lambda < 0$ $y = Ae^{kx} + Be^{-kx}$ $\lambda = 0$






5.3 Eigenvalue Problem Example

- We first check $\lambda > 0$, which has general solution:

$$y(x) = A \cos(kx) + B \sin(kx)$$

- Applying our first Boundary Condition

$$y(0) = A \cos 0 + B \sin 0 = A = 0$$


5.3 Eigenvalue Problem Example

- Since $A = 0$,

$$y(x) = B \sin(kx)$$

- Applying our next boundary condition,

$$y'(1) = Bk \cos(k) = 0$$

- Since we want a non-trivial answer, we do not want B or k to be 0. Therefore, we solve $\cos(k) = 0$, which has solutions as follows:

$$k = \frac{(2n + 1)\pi}{2}, \quad n = 0, 1, 2, \dots$$




5.3 Eigenvalue Problem Example

- By our earlier definitions:

$$\lambda_n = k_n^2 = \left(\frac{(2n+1)\pi}{2} \right)^2$$

and...

$$y_n(x) = \sin(k_n x) = \sin\left(\frac{(2n+1)\pi}{2} x\right)$$




5.3 Eigenvalue Problem Example

- We next check $\lambda = 0$, which has general solution:

$$y(x) = Ax + B.$$

- Applying the first boundary condition:

$$y(0) = 0 \Rightarrow B = 0, \text{ so } y(x) = Ax.$$

- Applying the second boundary condition:

$$y'(x) = A. \text{ Apply } y'(1) = 0 \Rightarrow A = 0.$$

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- So the only solution is $y = 0$

5.3 Eigenvalue Problem Example

- We next check $\lambda < 0$, which has general solution:

$$y(x) = Ae^{kx} + Be^{-kx}$$

- Using the first boundary condition,

$$y(0) = Ae^0 + Be^0 = A + B = 0 \quad \Rightarrow \quad B = -A$$

- Applying the second boundary condition:

$$y'(1) = 2Ak \cosh(k) = 0.$$

- This also implies that $A = 0$ because k and $\cosh(k)$ are not zero, which means that $A = B = 0$



Fourier Series

- (Nearly) any periodic function can be represented as an **infinite series of sin and cos functions**
- A Fourier Series **will always repeat periodically**, even if the modelled function is only defined on a certain interval

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Fourier Coefficients

- The coefficients of the series can be directly calculated using the orthogonality of sin and cos
- The integral **bounds are one period**
 - L is half of a period
- **For odd functions, $A_n = 0$**
- **For even functions, $B_n = 0$**

$$\begin{aligned}A_0 &= \frac{1}{2L} \int_0^{2L} f(x) dx \\A_n &= \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\B_n &= \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx\end{aligned}$$

Fourier Cosine Series

- The Fourier cosine series models the even extension of a function
- $f(-x) = f(x)$
- Note that for the cosine series **L is the full period** (because it's half the period when extended)

$$A_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

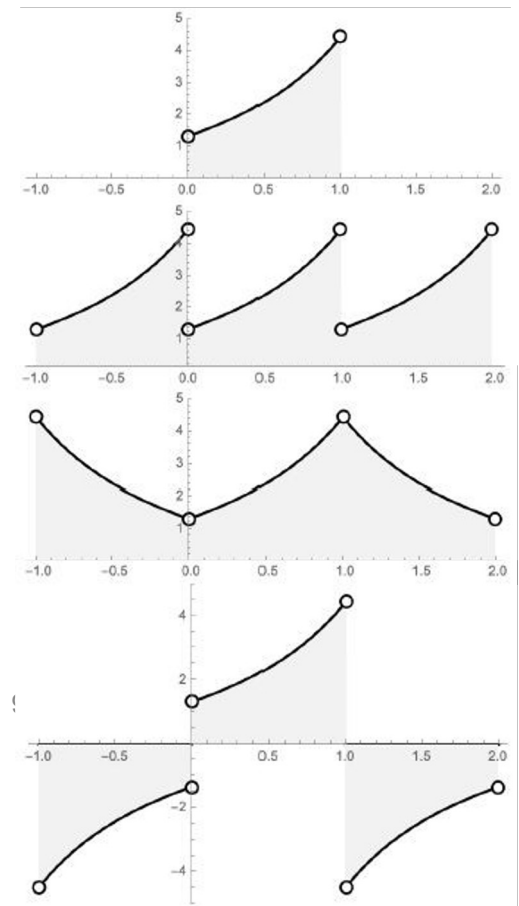
Fourier Sine Series

- The Fourier sine series models the odd extension of a function
- $f(-x) = -f(x)$
- Note that for the sine series **L is the full period** (because it's half the period when extended)

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$
$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Fourier Summary

- The “normal” Fourier series is the **periodic extension**
- The Fourier **cosine** series is the **even extension**
- The Fourier **sine** series is the **odd extension**





Fourier Convergence Theorem

Theorem 6.2.2. *Suppose that $f(x)$ is piecewise C^2 (twice differentiable) and $2L$ -periodic: $f(x + 2L) = f(x)$, with jump discontinuities at the points of discontinuity. Then the Fourier series*

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

converges to $f(x)$ at points of continuity of $f(x)$, and to $\frac{1}{2}(f(x^-) + f(x^+))$ at the jump discontinuities.

- If the modelled function is **continuous**, the series converges to the **function values**
- If the modelled function has a **discontinuity**, the series converges to the **average of the values at the jump**



Thanks for
Coming!

