Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 231 – Calculus II

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Apr 8, 7-9PM (Hriday, Tommy)

Session 2: Apr 26, 2-4PM (Sushrut, Amy)

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. You are given the power series

$$\sum_{n=1}^{\infty} \frac{(x-7)^n}{3^n(2n+1)}$$

.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

(a)

$$a_n = \frac{(x-7)^n}{3^n(2n+1)}, \qquad a_{n+1} = \frac{(x-7)^{n+1}}{3^{n+1}(2n+3)}$$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-7)^{n+1}}{3^{n+1}(2n+3)}\frac{3^n(2n+1)}{(x-7)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\frac{x-7}{3}\right| \\
&= \left|\frac{x-7}{3}\right| < 1 \\
&\implies |x-7| < 3 \implies \boxed{R = 3}
\end{aligned}$$

(b)

$$-3 < x - 7 < 3$$

$$4 < x < 10$$

$$\text{If } x = 4, \quad \sum_{n=1}^{\infty} \frac{(4-7)^n}{3^n(2n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3^n(2n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

This series at $x = 4$ converges by the alternating series test.

$$\text{If } x = 10, \quad \sum_{n=1}^{\infty} \frac{(10-7)^n}{3^n(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$$

This series at $x = 10$ diverges by the limit comparison test with $\frac{1}{n}$.
Therefore, the interval of convergence is $\boxed{[4, 10)}$.

2. Find the Maclaurin Series for:

(a)

$$f(x) = \frac{x}{1+4x^2}$$

(b)

$$f(x) = x^3 \cos\left(x^2\right)$$

(a) Notice power series:

$$x\left(\frac{1}{1+4x^2}\right) \to x\sum_{n=0}^{\infty}(-1)^n(4x^2)^n$$
$$= \boxed{\sum_{n=0}^{\infty}(-1)^n 4^n (x^{2n+1})}$$

(b)

$$\cos(x) \implies \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$
$$\cos\left(x^2\right) \implies \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!}$$
$$x^3\cos\left(x^2\right) \implies \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n)!}}$$

3. Find the first two non-zero terms of the Maclaurin Series for

(a) This function $f(x)$

$$f(x) = \frac{\sin(x^2)}{x}$$

(b) This function $g(x)$

$$g(x) = x\cos\big(x^2\big)$$

(c) Use parts (a) and (b) to write down the first two non-zero terms in the Maclaurin Series for the product $f(x)g(x)$.

(d) Use your work in parts (a) and (b) to evaluate:

$$\lim_{n\to 0} \frac{f(x) - g(x)}{x^5}$$

(a)

$$\sin\big(x^2\big) = x^2 - \frac{(x^2)^3}{3!} + \cdots = x^2 - \frac{x^6}{3!}$$

$$\implies f(x) \approx \frac{x^2 - \frac{x^6}{3!}}{x} = \boxed{x - \frac{x^5}{3!}}$$

(b)

$$\begin{aligned} g(x) &= x(\cos\big(x^2\big)) \\ &= x[1 - \frac{(x^2)^2}{2!} + \dots] \\ &= x[1 - \frac{x^4}{2!} + \dots] \\ &\approx \boxed{x - \frac{x^5}{2!}} \end{aligned}$$

(c)

$$f(x)g(x) \approx x^2 - \frac{x^6}{2!} - \frac{x^6}{3!} + \frac{x^{10}}{3!2!} \approx \boxed{x^2 - \frac{2}{3}x^6}$$

(d)

$$= \lim_{n\to 0} \frac{x - \frac{x^5}{3!} - x + \frac{x^5}{2!}}{x^5} = \lim_{n\to 0} \frac{x^5(\frac{1}{2} - \frac{1}{6})}{x^5} = \boxed{\frac{1}{3}}$$

4. The Maclaurin series expansion of $e^{\sin x}$ is:

(a) $1 + x - \frac{x^2}{2} + \frac{x^4}{12} - \cdots$

(b) $1 - x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

(c) $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

(d) $1 + x + \frac{x^2}{2} - \frac{x^4}{12} + \cdots$

Let $f(x) = e^{\sin x}$.

$$\begin{aligned}
f'(x) &= e^{\sin x} \cos x = f(x) \cos x \\
f''(x) &= f'(x) \cos x - f(x) \sin x \\
f'''(x) &= f''(x) \cos x - f'(x) \sin x - (f(x) \cos x + f'(x) \sin x) \\
&= f''(x) \cos x - 2f'(x) \sin x - f'(x) \\
f^4(x) &= f'''(x) \cos x - f''(x) \sin x - 2(f'(x) \cos x + f''(x) \sin x) - f''(x) \\
&= f'''(x) \cos x - 3f''(x) \sin x - 2f'(x) \cos x - f''(x)
\end{aligned}$$

Calculating higher order derivatives at $x = 0$,

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= 1 \times 1 = 1 \\
f''(0) &= 1 - 0 = 1 \\
f'''(0) &= 1 - 0 - 1 = 0 \\
f^4(0) &= 0 - 0 - 2 - 1 = -3 \\
f^5(0) &= f^4(0) \cos 0 - 3f''(0) \cos 0 - 2f''(0) \cos 0 - f'''(0) \\
&= -3 - 3 - 2 - 0 = -8
\end{aligned}$$

Using Maclaurin's expansion for infinite series,

$$\begin{aligned}
f(x) &= f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \\
&\quad \frac{x^4}{4!} f^4(0) + \frac{x^5}{5!} f^5(0) + \frac{x^6}{6!} f^6(0) + \ldots \\
e^{\sin x} &= 1 + x + \frac{x^2}{2!} \times 1 + 0 + \frac{x^4}{4!} \times (-3) + \frac{x^5}{5!} \times (-8) \ldots \\
&= \boxed{1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} \ldots}
\end{aligned}$$

5. The Maclaurin Series for $f(x)$ is shown below. Find $f^{(10)}(0)$

$$\sum_{n=1}^{\infty} \frac{(x)^{2n}}{4n^2 9^n}$$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$$\frac{f^{(10)}(0)}{10!} = c_{10} = \frac{1}{4 \cdot 5^2 \cdot 9^5} = \frac{1}{100 \cdot 9^5}$$

$$f^{(10)}(0) = \frac{10!}{100 \cdot 9^5} = \frac{9!}{10 \cdot 9^5} = \boxed{\frac{8!}{10 \cdot 9^4}}$$

6. Find the correct formula for the Taylor Series of $f'(x)$, the derivative of $f(x)$, centered at a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} n(x-a)^{n-1} = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$$

$$k = n-1 \implies n = k+1$$

$$\boxed{f'(x) = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(a)}{k!}(x-a)^k}$$

7. Find the area outside of the region r $= 3 + 2sin(\theta)$ and inside r $= 2$.
Find the points of intersection:

$$3 + 2sin(\theta) = 2$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Apply the formula:

$$A = \int_a^b \frac{1}{2}(r_{outer}^2 - r_{inner}^2)d\theta$$

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(2^2 - (3 + 2sin(\theta))^2)d\theta$$

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(-5 - 12sin(\theta) - 4sin(\theta)d\theta$$

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}(-7 - 12sin(\theta) + 2cos(2\theta))d\theta$$

$$A = \frac{1}{2}(7\theta + 12cos(\theta) + sin(2\theta)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}}$$

$$A = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}$$

8. Match the parametric equations with the graphs

(i) $x = t^4 - t + 1$
$y = t^2$

(ii) $x = t^2 - 2t$
$y = \sqrt{t}$

(iii) $x = sin(2t)$
$y = sin(t + sin(2t))$

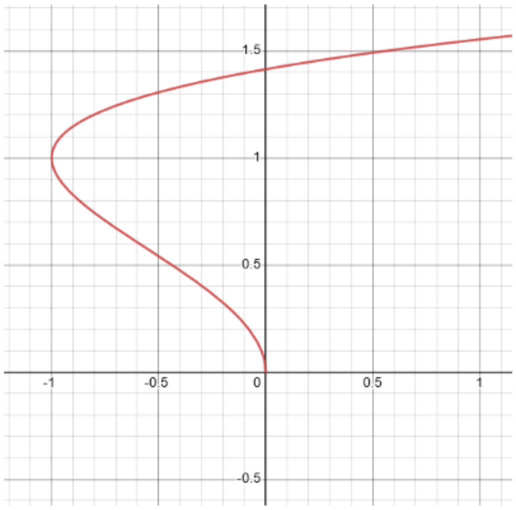
(iv) $x = cos(5t)$
$y = sin(2t)$

(v) $x = t + sin(4t)$
$y = t^2 + cos(3t)$

(vi) $x = \frac{sin(2t)}{4+t^2}$
$y = \frac{cos(2t)}{4+t^2}$

(i) $\Longrightarrow$ Graph E
(ii) $\Longrightarrow$ Graph A

$(iii) \implies$ Graph B
$(iV) \implies$ Graph D
$(v) \implies$ Graph F
$(vi) \implies$ Graph C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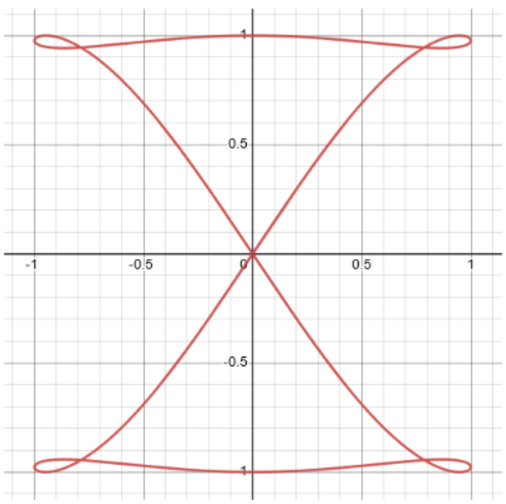
1
-0.5

(a)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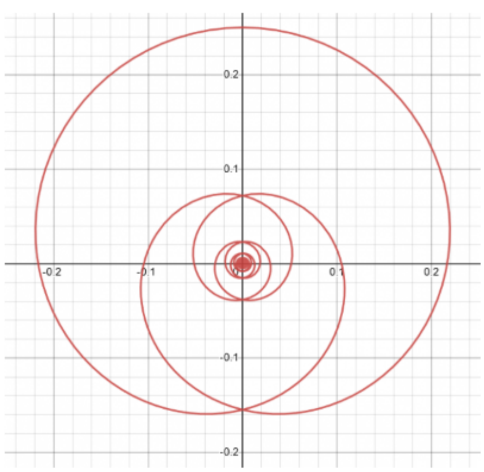
-1

(b)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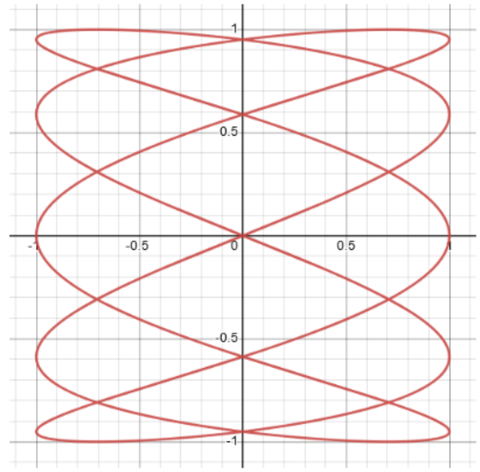
(c)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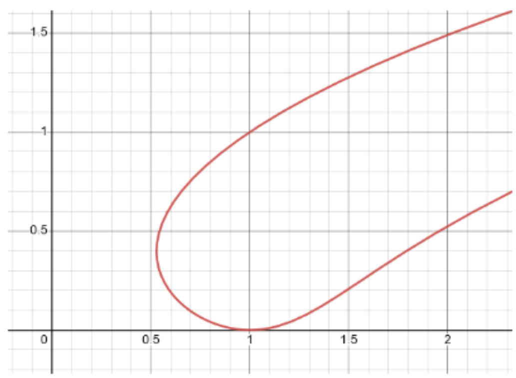
(d)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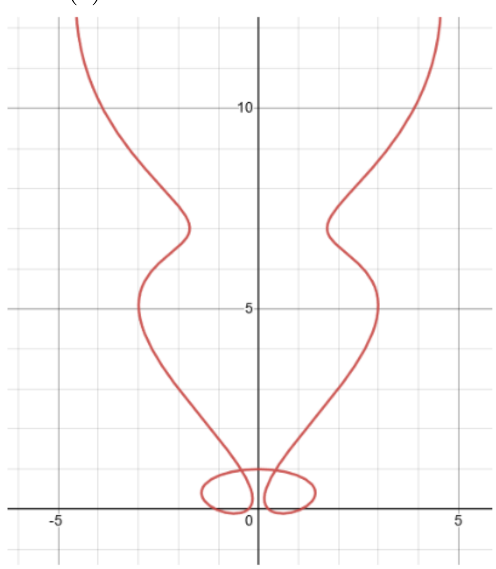
1
1.5
2

(e)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5

(f)