The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 4 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

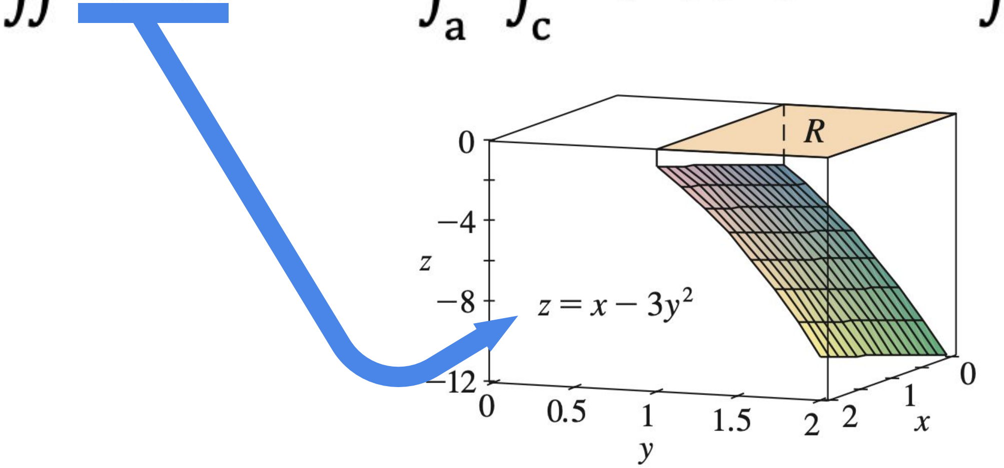
The queue contains the worksheet and the solution to this review session

Fubini's Theorem

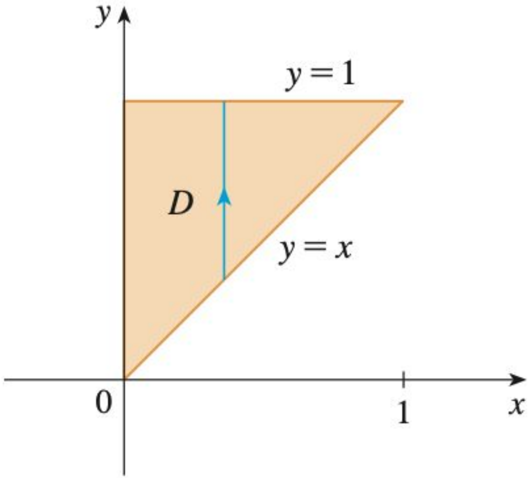
- If f(x,y) is continuous on the rectangle

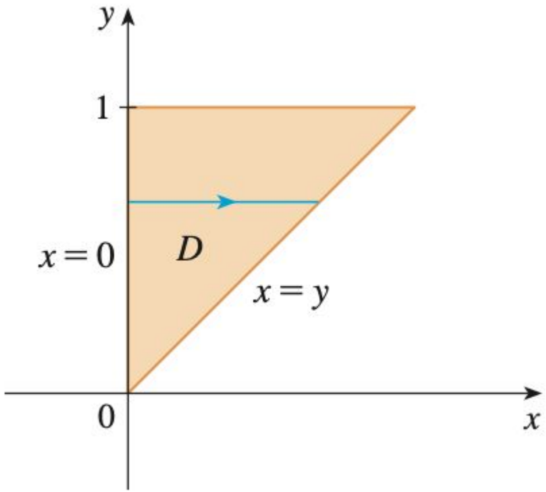
$$R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$$

$$\iint f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$



Double Integral Over a General Region





- Integrate dy from y=x to y=1
- Then integrate dx

- Integrate dx from x=0 to x=y
- Then integrate dy

Center of Mass

- The x, y coordinates of the center of mass for an object that has a density function ρ(x,y)

$$\bar{\mathrm{x}} = \frac{1}{\mathrm{m}} \iint x \cdot \rho(x, y) dA \qquad \bar{\mathrm{y}} = \frac{1}{\mathrm{m}} \iint y \cdot \rho(x, y) dA$$

, where mass is calculated as $\mathrm{m} = \iint \rho(x, y) dA$

Triple Integral

- **Let *E* be the solid contained under the plane 2x + 3y + z = 6 in the first octant. Compute the following:**

$$\iiint_E 2x \, dV$$

Triple Integral-Cont'd

- **Let *E* be the solid contained under the plane 2x + 3y + z = 6 in the first octant. Compute the following:**

$$\iiint_E 2x\,dV = \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3y} 2x\,dzdydx = \int_0^3 \int_0^{2-2x/3} 2x(6-2x-3y)dydx$$

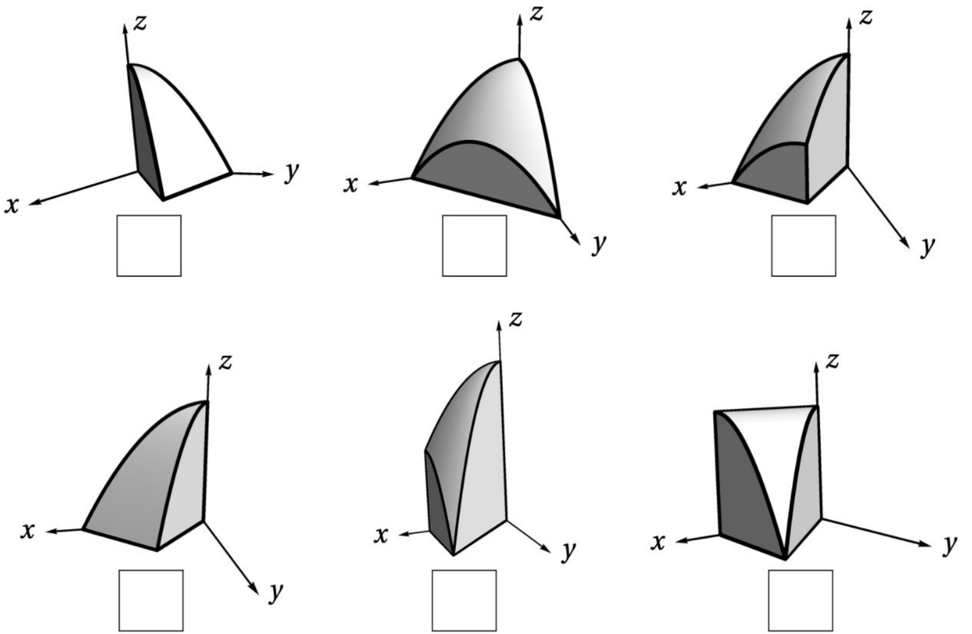
$$= \int_0^3 12x\left(2-\frac{2x}{3}\right) - 4x^2\left(2-\frac{2x}{3}\right) - 3x\left(2-\frac{2x}{3}\right)^2 dx = 9$$

Example Question #1

- **Match the integrals to their corresponding solid regions:**

(A) $\displaystyle\int_0^1 \int_y^1 \int_0^{2-x^2-y^2} f(x,y,z)\, dz\, dx\, dy$

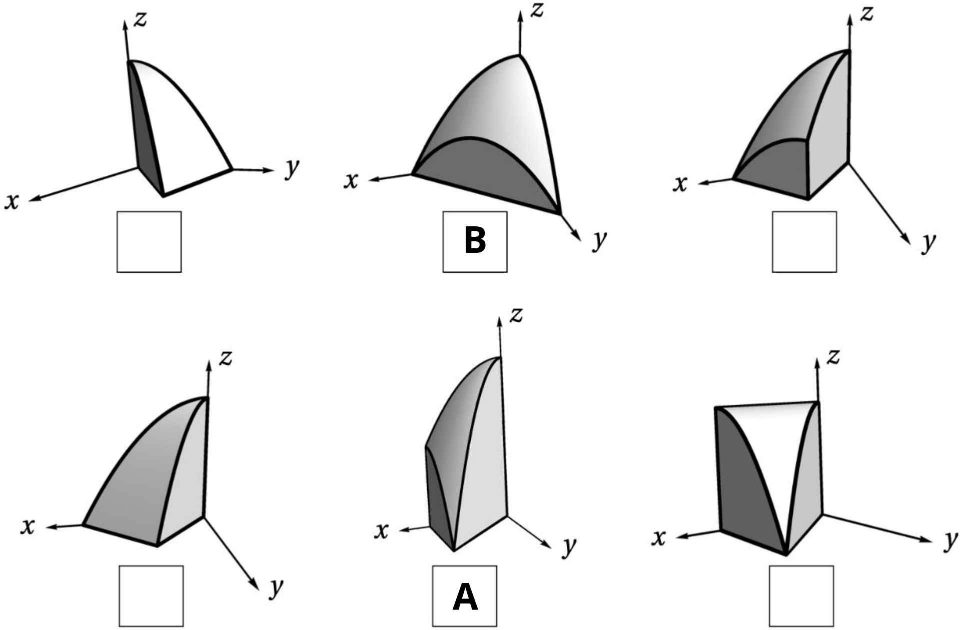
(B) $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} g(x,y,z)\, dz\, dy\, dx$



Example Solution #1

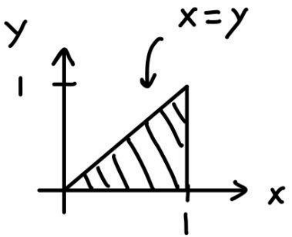
(A) $$\int_0^1 \int_y^1 \int_0^{2-x^2-y^2} f(x, y, z)\, dz\, dx\, dy$$

(B) $$\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} g(x, y, z)\, dz\, dy\, dx$$

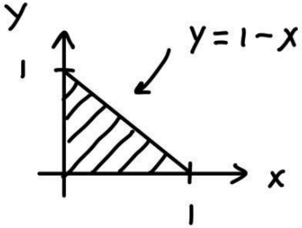


base

A



B



Polar Coordinates

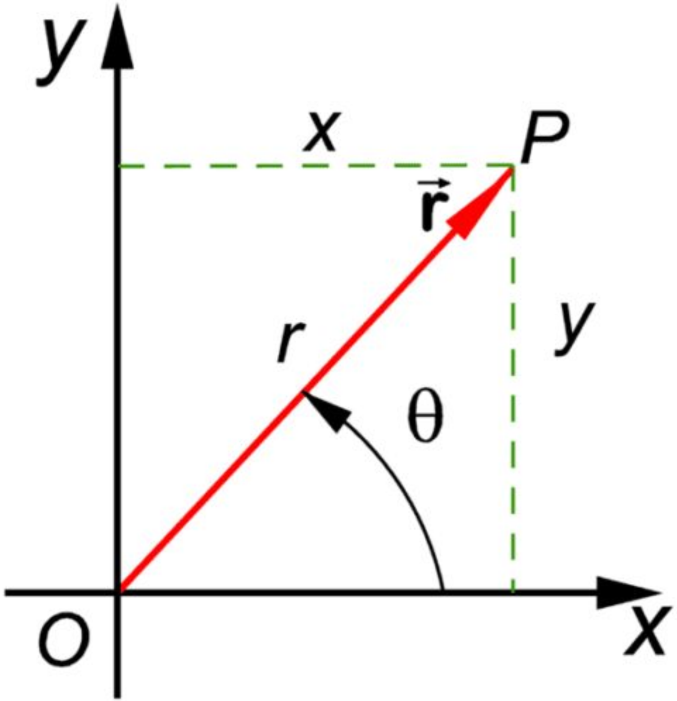
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\theta = arctan\left(\frac{y}{x}\right)$$

$$dA = r\,dr\,d\theta$$



https://magoosh.com/hs/ap-calculus/2017/ap-calculus-bc-review-polar-functions/

Cylindrical Coordinates

- **Cylindrical coordinate is just an extension of polar coordinate to three dimension**

$$x = r\cos\theta$$

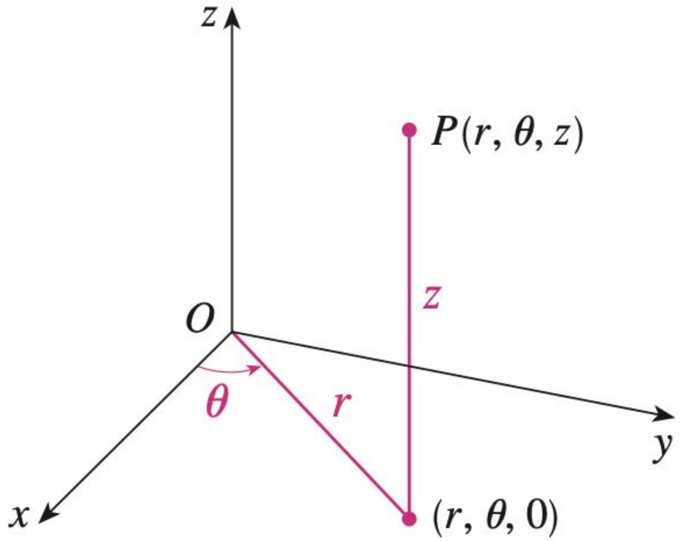
$$y = r\sin\theta$$

$$z = z$$

$$r^2 = x^2 + y^2$$

$$\theta = arctan\left(\frac{y}{x}\right)$$

$$dV = rdzdrd\theta$$



Sketch of a point in R^3

Spherical Coordinates

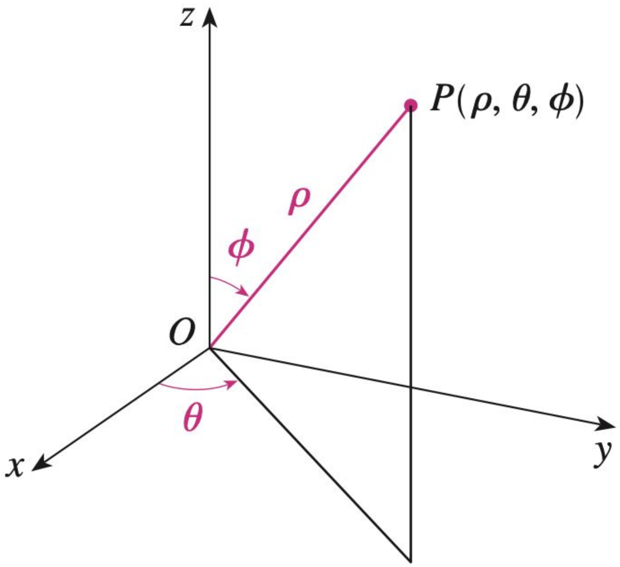
$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$$



Sketch of a point in R^3

Surface Area

- **The area of the surface A(S) with equation z=f(x,y) can be calculated as:**

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Change of Variables Using Jacobian Matrix

- **If there is a transformation such that x=g(u,v) and y=h(u,v), then:**

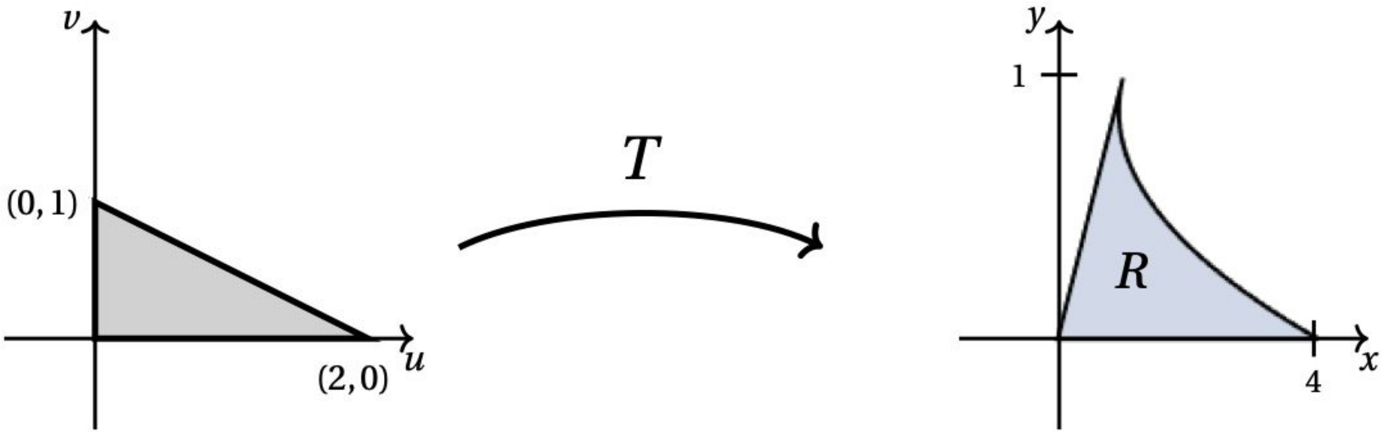
$$\iint_R f(x,y)\,dA = \iint_S f[g(u,v),h(u,v)] \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| d\bar{A}$$

, where the Jacobian Matrix is calculated as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

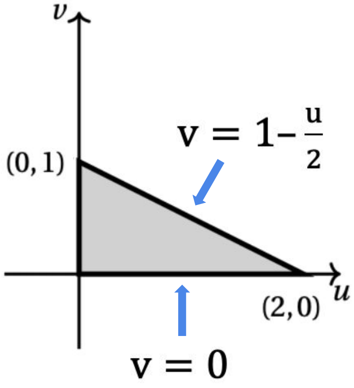
Example Question #2

- **Set up the integral to calculate the area of R with the transformation T(u,v) = (u^2+v, v).**



Example Solution #2

- **Set up the integral to calculate the area of R with the transformation $T(u,v) = (u^2+v, v)$.**



$$0 \leq v \leq 1 - \frac{u}{2} \qquad 0 \leq u \leq 2$$

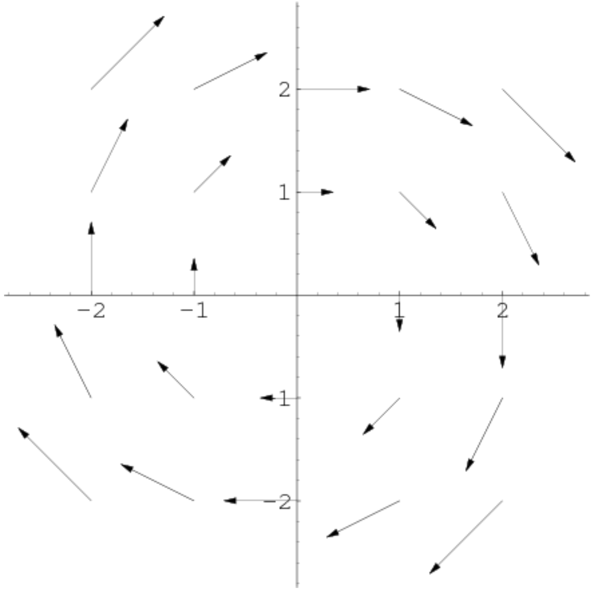
Jacobian: $\det \begin{bmatrix} 2u & 1 \\ 0 & 1 \end{bmatrix} = 2u$

Integral: $\int_0^2 \int_0^{1-u/2} 2u \, dv du$

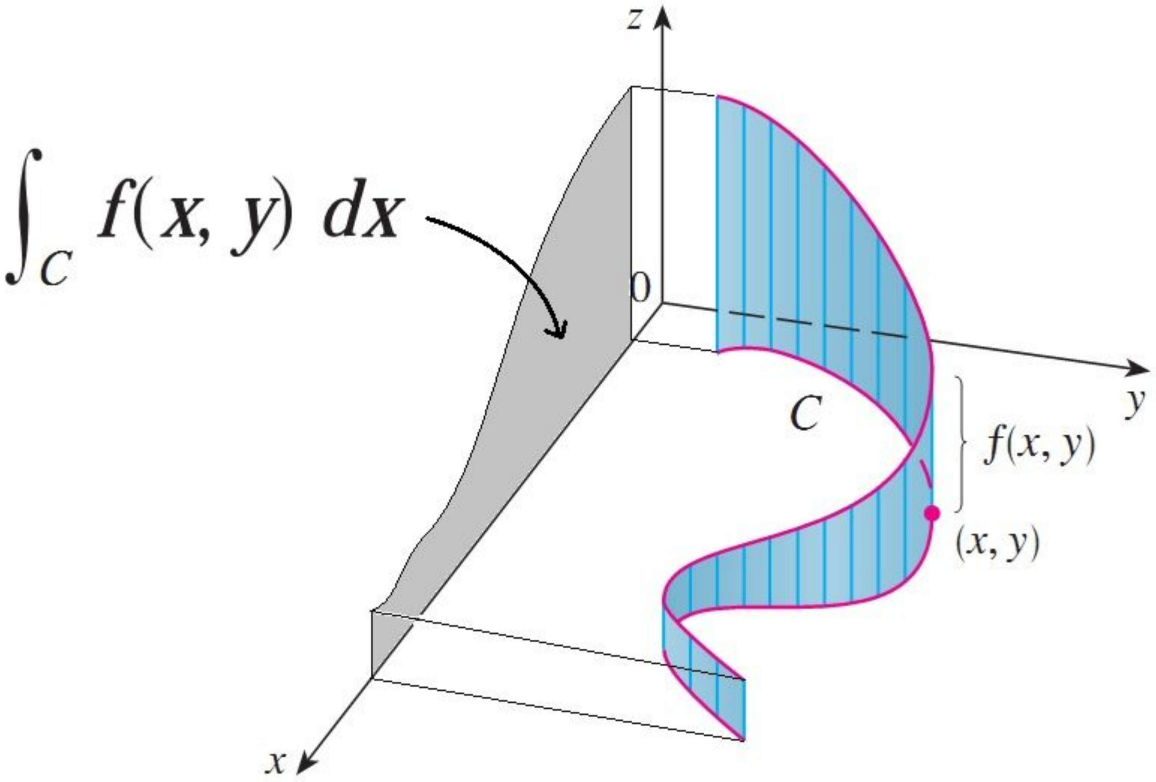
Vector Field

- A function that assigns a vector **F** to each point in 2D or 3D space.
- Takes in a point and "spits out" a vector

$$\vec{\mathbf{F}}(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}.$$



Intro Line Integrals



Line Integrals on Scalar Value Functions

- f takes in a vector and spits out a scalar
- ds is infinitesimal change on the curve C
- r(t) is a vector valued function which represents the curve C.

$$\int_C f(x,y)\,ds$$

$$\oint f(r)ds = \int_a^b f[r(t)]\,|r'(t)|\,dt$$

$$\int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Line integrals of scalar function Part 2

$$\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$$

Line integral of f over a curve c with respect to a change in x,
NOT arc length

Line Integrals on Vector Value Functions

- F takes in a vector and spits out a vector
- r(t) is a vector function over time, which represents the curve C that we are integrating over

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\downarrow$$

$$\int_C \langle g(x,y), h(x,y) \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle\, dt$$

$$\downarrow$$

$$\int_C g\, dx + h\, dy$$

Fundamental theorem of line integrals

Suppose that C is a **smooth** curve given by $\vec{r}\,(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}\,(b)\right) - f\left(\vec{r}\,(a)\right)$$

Compare to FTC:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$