MATH 231 Exam Review

Midterm 03

Sequences

Sequence: Just a list of the numbers

- Pattern Recognition!

- To test whether convergent or divergent take the limit, if the limit exists: convergent, if not! It is divergent.
  - If the sequence is a function, take the limit of the function
  - If cannot take limit, Squeeze Theorem!

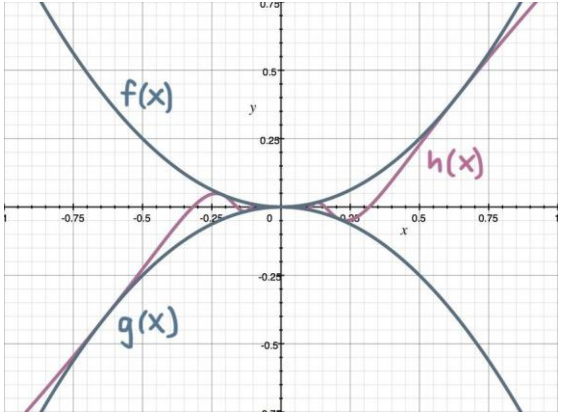
Useful Squeeze Theorem

- $\lim_{x \to \infty} \left(\frac{\sin(x)}{x}\right) = 0$
- $\lim_{x \to 0} \left(\frac{\sin(ax)}{x}\right) = a$
- $\lim_{x \to 0} \left(\frac{\sin(x)}{x}\right) = 1$
- $\lim_{x \to 0} \left(\frac{\cos(x)-1}{x}\right) = 0$

$$f(x) \leq h(x) \leq g(x)$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

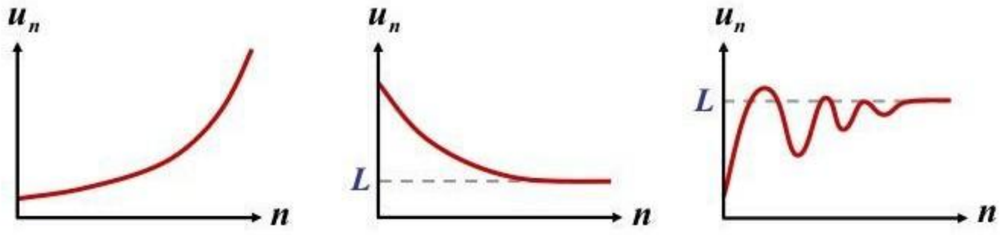
$$\lim_{x \to c} h(x) = L$$



Sequences

Convergence and Divergence.

Some sequences never stop increasing, while others eventually settle at a particular number.



If the numbers in a sequence continue to get further and further apart, the sequence **diverges**.

If a sequence **tends towards a limit**, it is described as **convergent**.

Sequence Convergence

- Convergence:
  - Increasing
    - if all $a_n < a_{n+1}$
  - Decreasing
    - if all $a_n > a_{n+1}$
  - Bounded from Below
    - If there existed a number m such that $\mathrm{m} \leq a_{n+1}$
  - Bounded from Above
    - If there existed a number M such that $\mathrm{M} \geq a_{n+1}$
  - If a sequence is bounded (from above and below) and monotonic (increasing or decreasing only), then it is convergent
    - If is not both of these, does not necessarily mean it is divergent

Series

- Series: The sum of a sequence.
  - If a series converges, then the sequence must converge as well.
  - **However:** If sequence converges, then the series may or may not converge.
  - Σa_n converges if the limit of the series converges.
- Geometric series:
  - $\Sigma ar^{k-1} = a + ar + ar^2 + ar^3 \ldots .$
  - Will converge if |r| < 1
- Other techniques:
  - Evaluate the partial sums (first bit of sums) of a series and see how the series behaves
- If Σa_k converges, then $\lim_{x \to \infty} a_n = 0$

Integral Test

- Suppose $a_n = f(n)$

  If converges $\int_1^\infty f(x)dx$ then $\sum_{n=1}^\infty a_n$ converges

  If diverges $\int_1^\infty f(x)dx$ then $\sum_{n=1}^\infty a_n$ diverges

- P-Test

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

When does this series converge?

When p > 1!

Comparison Test

- If given two series and know the convergence or divergence of one:

If $\sum_{n=1}^{\infty} b_n$ is convergent and $b_n \geq a_n$ then $\sum_{n=1}^{\infty} a_n$ converges

If $\sum_{n=1}^{\infty} b_n$ is divergent and $b_n \leq a_n$ then $\sum_{n=1}^{\infty} a_n$ diverges

- Limit Comparis on Test:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and c is finite and > 0:

Either both series converge or both series diverge

Alternating Series

- The series alternates between positive and negative!
- Convergence:
  - Terms are decreasing, $b_n >= b_{n+1}$
  - $\lim_{n \to \infty} b_n = 0$
- Absolute and Conditional Convergence
  - Absolute Convergence: The absolute value of a series is convergent
  - Conditional Convergence: The absolute value of a series is diverges, but the alternating series converges
  - If a series is absolutely convergent, then it is convergent

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Conditional or Absolute convergence?

Conditionally Convergent!

Ratio Test

- Take the limit of absolute value of the ratio of the n^{th} term and the $n^{th}+1$ term

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

- If L < 1: Absolutely Convergent
- If L > 1 or L = ∞: Divergent
- If L = 1: Inconclusive

Root Test

- Take the limit of n^{th} root of the absolute value of the n^{th} term

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

- If $L < 1$: Absolutely Convergent
- If $L > 1$ or $L = \infty$: Divergent
- If $L = 1$: Inconclusive

Putting It All Together

A general order for how you might want to go by solving problems:

1. Test for Divergence
2. p-Series Test
3. Geometric Series Test
4. Comparison Test
5. Alternating Series Test
6. Ratio Test
7. Root Test
8. Integral Test

Putting It All Together

1. **Check divergence with limit**
2. Look for easy P-Test/Geometric
3. Inspection

TEST	SERIES	CONVERGES IF...	DIVERGES IF...	COMMENTS
nth Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$	n/a	$\lim_{n\to\infty} \neq 0$	should be first test used. Inconclusive if limit = 0.
Geometric Series Test	$\sum_{n=1}^{\infty} a_n r^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	use if there is a "common ratio" $S_n = \frac{a}{1-r}$
P-Series Test	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	harmonic series when p=1. Useful for comparison tests.
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(x)$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	$f(x)$ must be continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges	to show convergence, find a larger series. to show divergence, find a smaller series.
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ diverges	apply l'hospital's rule if necessary; inconclusive if limit equals 0 or ∞
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$a_{n+1} \leq a_n$, $\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	must prove that the limit equals 0 and that terms are decreasing
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	test fails if : $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	test fails if: $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$