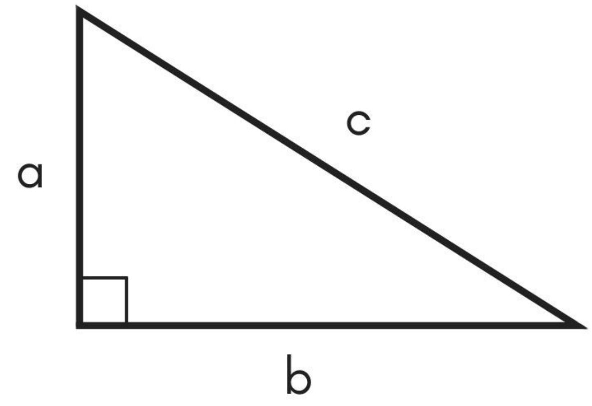
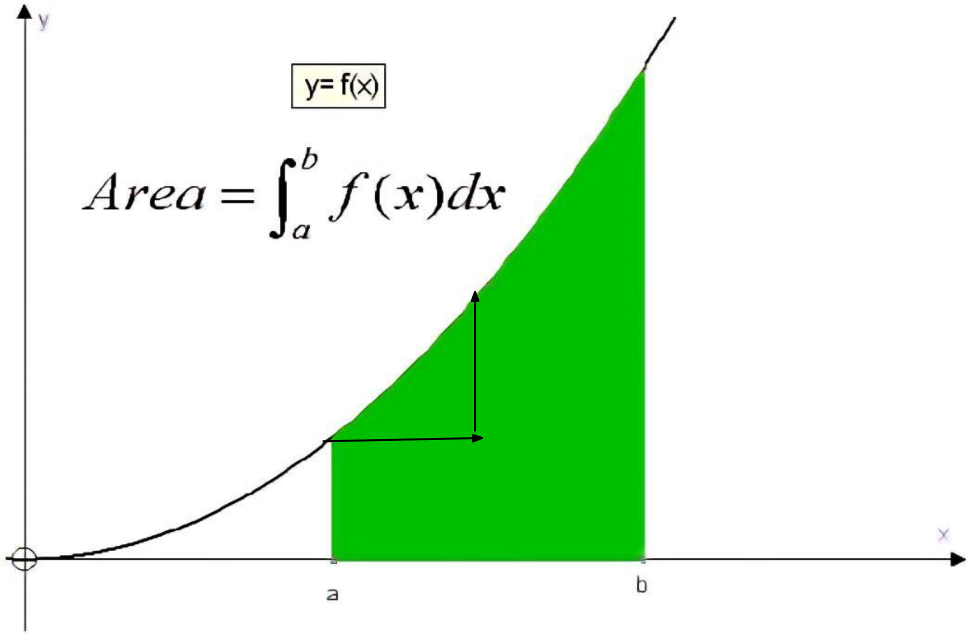
MATH 231 Exam Review

Midterm 02

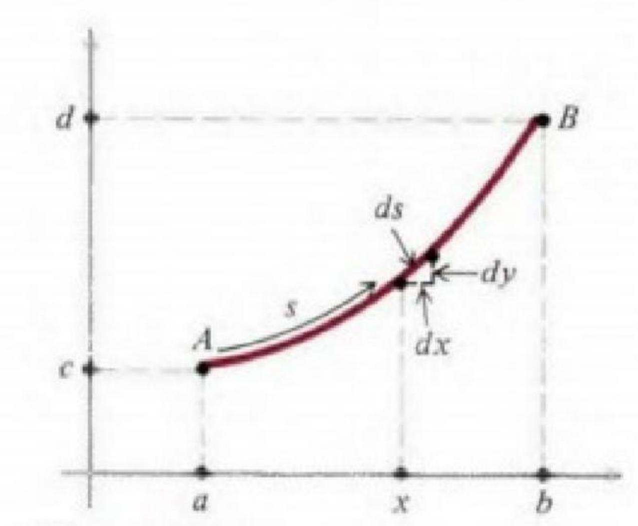
Arc Length



$$a^2 + b^2 = c^2$$



Arc Length



$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

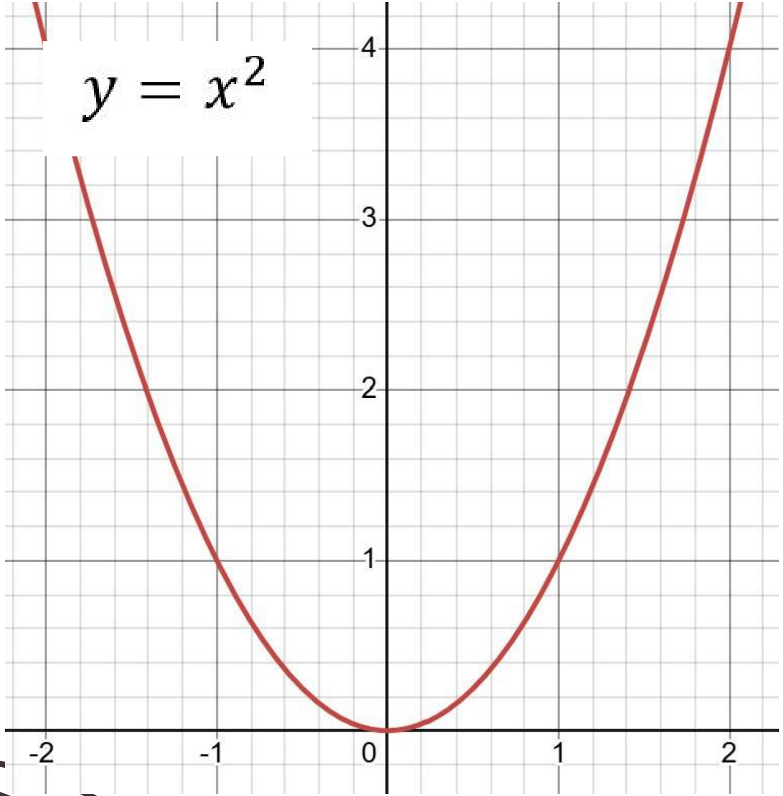
$$= \sqrt{\left(1 + \frac{dy^2}{dx^2}\right) dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$\text{length of arc } AB = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

When to use which formula… and how to go about each one

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx \quad \Longrightarrow \quad a \leq x \leq b \qquad y = x^2$$

$$L = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2} dy \quad \Longrightarrow \quad a \leq y \leq b \qquad x = \sqrt{y}$$

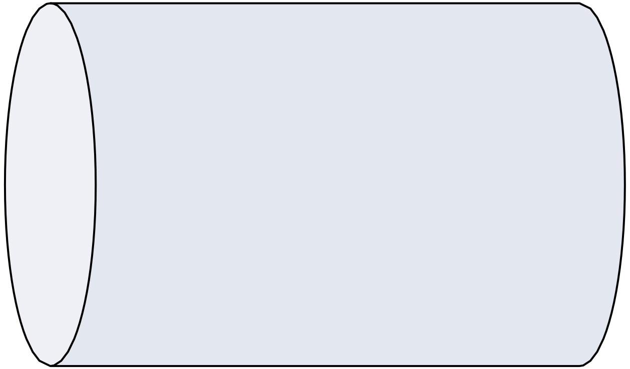


$$\int_0^2 \sqrt{1 + (2x)^2}\,dx = \int_0^4 \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

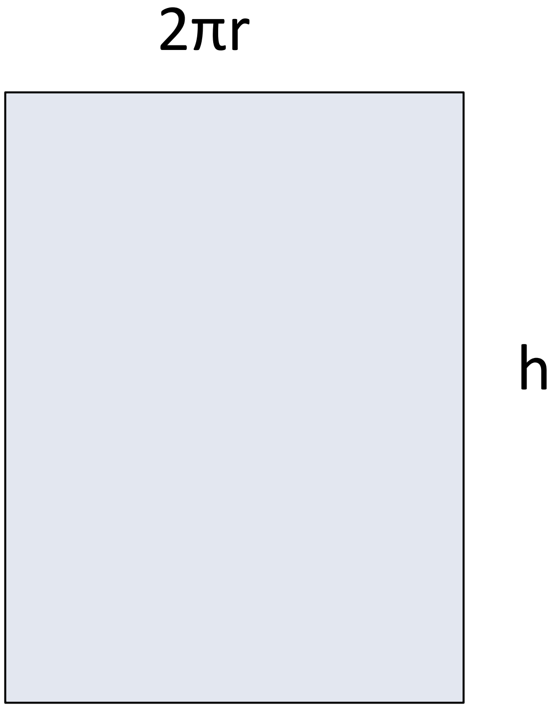
General steps for solving arc length problems:

1. Write down formula that makes the most sense based on what you are given in the problem
2. Find the derivative
3. Set up the integrand and solve

Surface Area of a Revolution

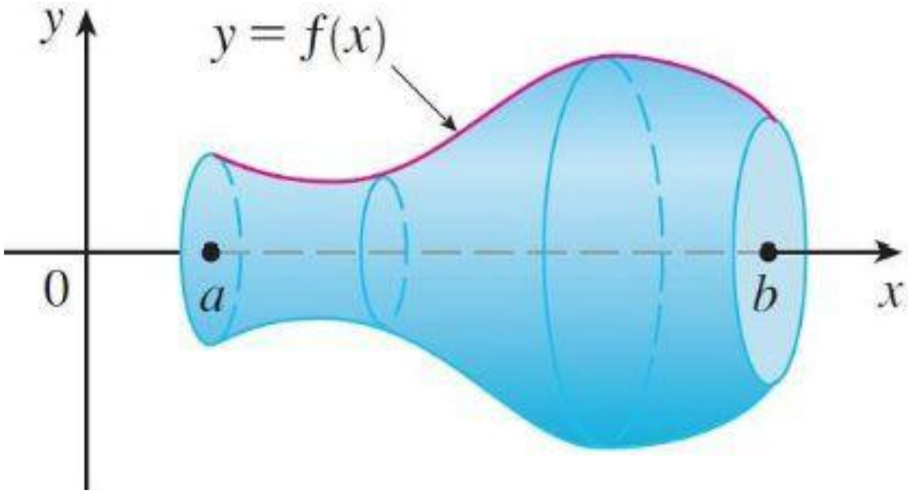


$$A = 2\pi r \times h$$



Surface Area of Revolution

We apply the same knowledge to more complex shapes, the arc length will be the ‘h’ and then the given function will be your circumference.



Surface Area of Revolution

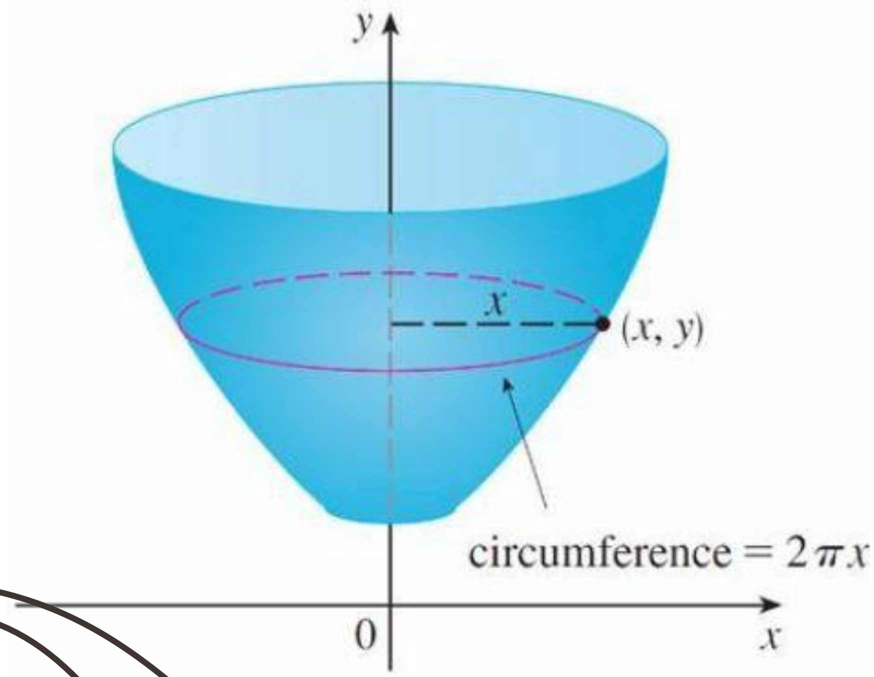
$$S = \int 2\pi y \, ds$$ ⟹ Around x-axis

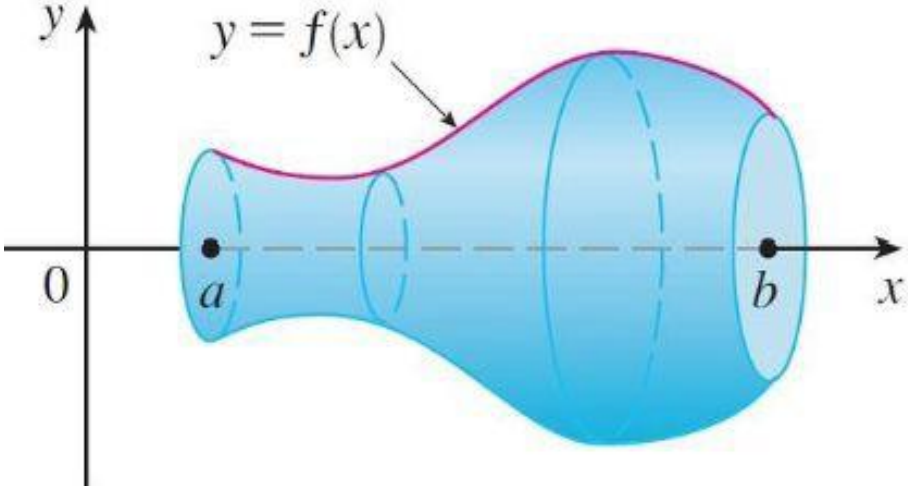
$$S = \int 2\pi x \, ds$$ ⟹ Around y-axis

$$ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2 dx}$$

$$ds = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2 dy}$$

Visualizing what is going on





Moments and Center of Mass

Center of mass:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Moments about axis:

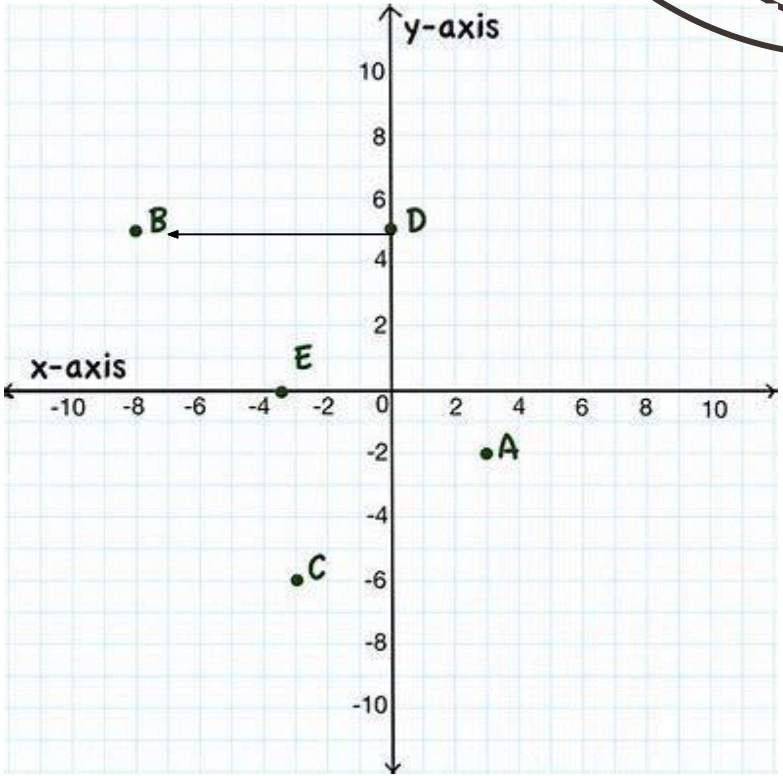
$$M_y = \sum m_i x_i$$

$$M_x = \sum m_i y_i$$

Center of mass coordinates:

$$\bar{x} = \frac{M_y}{m}$$

$$\bar{y} = \frac{M_x}{m}$$



Uniform density

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2 dx$$

If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, the centroid of $\mathcal{R}$ is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}dx$$

Sequences

Sequence: Just a list of the numbers

- Pattern Recognition!

- To test whether convergent or divergent take the limit, if the limit exists: convergent, if not! It is divergent.

  - If the sequence is a function, take the limit of the function

  - If cannot take limit, Squeeze Theorem!

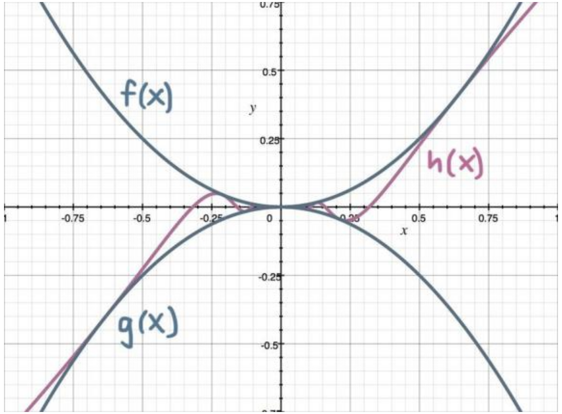
Useful Squeeze Theorem

- $\lim_{x\to\infty}\left(\frac{\sin(x)}{x}\right) = 0$
- $\lim_{x\to 0}\left(\frac{\sin(ax)}{x}\right) = a$
- $\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = 1$
- $\lim_{x\to 0}\left(\frac{\cos(x)-1}{x}\right) = 0$

$$f(x) \leq h(x) \leq g(x)$$

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$$

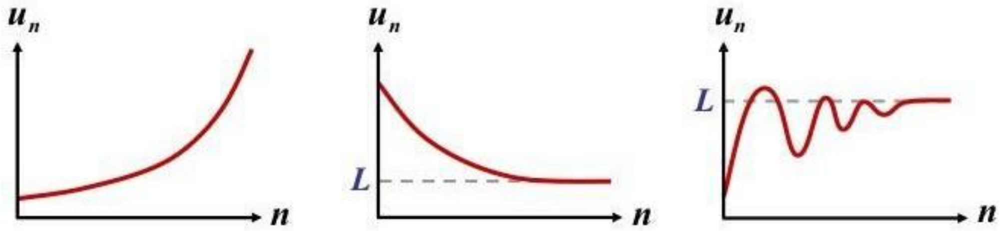
$$\lim_{x\to c} h(x) = L$$



Sequences

Convergence and Divergence.

Some sequences never stop increasing, while others eventually settle at a particular number.



If the numbers in a sequence continue to get further and further apart, the sequence diverges.

If a sequence tends towards a limit, it is described as convergent.

Sequence Convergence

- Convergence:
  - Increasing
    - if all $a_n < a_{n+1}$
  - Decreasing
    - if all $a_n > a_{n+1}$
  - Bounded from Below
    - If there existed a number m such that $\mathrm{m} \leq a_{n+1}$
  - Bounded from Above
    - If there existed a number M such that $\mathrm{M} \geq a_{n+1}$
  - If a sequence is bounded (from above and below) and monotonic (increasing or decreasing only), then it is convergent
    - If is not both of these, does not necessarily mean it is divergent

Series

- Series: The sum of a sequence.
  - If a series converges, then the sequence must converge as well.
  - **However:** If sequence converges, then the series may or may not converge.
  - Σa_n converges if the limit of the series converges.
- Geometric series:
  - $\Sigma ar^{k-1} = a + ar + ar^2 + ar^3 \ldots.$
  - Will converge if $|r| < 1$
- Other techniques:
  - Evaluate the partial sums (first bit of sums) of a series and see how the series behaves
- If Σa_k converges, then $\lim_{x \to \infty} a_n = 0$

Integral Test

- Suppose $a_n = f(n)$

  If converges $\int_1^\infty f(x)dx$ $\sum_{n=1}^\infty a_n$ n

converges

$\int_1^\infty f(x)dx$ $\sum_{n=1}^\infty a_n$

  If diverges then diverges

- P-Test

When does this series converge?

When $p > 1$!

$\sum_{n=1}^\infty \frac{1}{n^p}$

Comparison Test

- If given two series and know the convergence or divergence of one:

If $\sum_{n=1}^{\infty} b_n$ is convergent and $b_n \geq a_n$ then $\sum_{n=1}^{\infty} a_n$ converges

If $\sum_{n=1}^{\infty} b_n$ is divergent and $b_n \leq a_n$ then $\sum_{n=1}^{\infty} a_n$ diverges

- Limit Comparis on Test:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and c is finite and > 0:

Either both series converge or both series diverge

Alternating Series

- The series alternates between positive and negative!
- Convergence:
  - Terms are decreasing, $b_n >= b_{n+1}$
  - $\lim_{n \to \infty} b_n = 0$
- Absolute and Conditional Convergence
  - Absolute Convergence: The absolute value of a series is convergent
  - Conditional Convergence: The absolute value of a series is diverges, but the alternating series converges
  - If a series is absolutely convergent, then it is convergent

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Conditional or Absolute convergence?

Conditionally Convergent!

Ratio Test

- Take the limit of absolute value of the ratio of the n^{th} term and the $n^{th}+1$ term

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

- If $L < 1$: Absolutely Convergent
- If $L > 1$ or $L = \infty$: Divergent
- If $L = 1$: Inconclusive

Root Test

- Take the limit of n^{th} root of the absolute value of the n^{th} term

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

- If L < 1: Absolutely Convergent
- If L > 1 or L = ∞: Divergent
- If L = 1: Inconclusive

Putting It All Together

A general order for how you might want to go by solving problems:

1. Test for Divergence
2. p-Series Test
3. Geometric Series Test
4. Comparison Test
5. Alternating Series Test
6. Ratio Test
7. Root Test
8. Integral Test

Putting It All Together

1. **Check divergence with limit**
2. Look for easy P-Test/Geometric
3. Inspection

TEST	SERIES	CONVERGES IF...	DIVERGES IF...	COMMENTS
nth Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$	n/a	$\lim_{n\to\infty} \neq 0$	should be first test used. Inconclusive if limit = 0.
Geometric Series Test	$\sum_{n=1}^{\infty} a_n r^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	use if there is a "common ratio" $S_n = \frac{a}{1-r}$
P-Series Test	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	harmonic series when p=1. Useful for comparison tests.
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(x)$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	$f(x)$ must be continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges	to show convergence, find a larger series. to show divergence, find a smaller series.
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ diverges	apply l'hospital's rule if necessary; inconclusive if limit equals 0 or ∞
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$a_{n+1} \leq a_n$, $\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	must prove that the limit equals 0 and that terms are decreasing
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$	$\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$	test fails if : $\lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	test fails if: $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1$