



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 231 – Calculus II

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: March 9th, 7:00 - 8:50 pm (Hriday, Jaylin, Amy)

Session 2: March 29th, 6:00 - 7:50 pm (Tommy, Sushrut, and Yash)

Can't make it to a session? Here's our schedule by course:

<https://care.grainger.illinois.edu/tutoring/schedule-by-subject>

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: <https://queue.illinois.edu/q/queue/844>
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Determine whether the integral converges or diverges. If so, what value does the integral converge to?

$$\int_5^{\infty} \frac{1}{n(\ln(n))^2}$$

$$= \int_5^{\infty} \frac{1}{x(\ln(x))^2} dx \quad [u = \ln(x), du = \frac{1}{x}]$$

$$= \frac{-1}{\ln(x)} \Big|_5^{\infty} = \frac{1}{\ln(5)} \rightarrow \text{so the integral converges.}$$

2. Test the following improper integral for convergence. If it converges, find the value

$$\int_0^{\infty} e^{-\frac{4}{3}y} dy$$

- (a) Diverges
 (b) Converges, value is $-\frac{3}{4}$
 (c) Converges, value is $\frac{4}{3}$
 (d) Converges, value is $\frac{3}{4}$
 (e) None of the above

This is an improper integral, so we need to check for convergence.

$$\lim_{t \rightarrow \infty} \int_0^t e^{-\frac{4}{3}y} dy$$

$$\lim_{t \rightarrow \infty} \left(\frac{-3}{4} e^{-\frac{4}{3}y} \right) \Big|_0^t$$

$$\lim_{t \rightarrow \infty} \left(\left(\frac{-3}{4} e^{-\frac{4}{3}t} \right) - \left(\frac{-3}{4} e^0 \right) \right)$$

$$\lim_{t \rightarrow \infty} \frac{-3}{4e^{\frac{4}{3}t}} + \frac{3}{4}$$

$$\frac{-3}{\infty} + \frac{3}{4} = \boxed{\frac{3}{4}}$$

3. Determine if the following integral is convergent or divergent:

$$\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$$

To solve this we must use the comparison test. Note that the numerator is bounded by:

$$0 \leq \cos^2 x \leq 1$$

Therefore the behavior of the denominator will determine the convergence/divergence. The integral

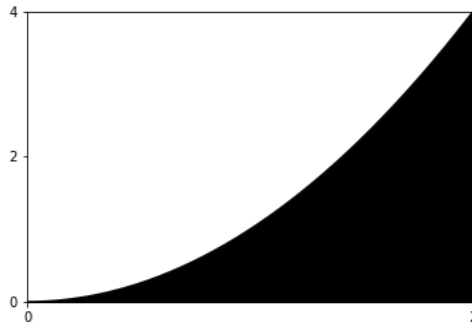
$$\int_2^{\infty} \frac{1}{x^2} dx$$

must converge via the p -test since $p = 2 > 1$. Putting these two facts together, we see that

$$\frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}$$

, as the numerator of $\cos^2 x$ will always be ≤ 1 . Since $\int_2^{\infty} \frac{1}{x^2} dx$ converges, then our original integral $\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$ must converge.

4. Find the M_x , M_y , and the centroid of $y = x^2$ with density λ on $x \in [0, 2]$.



$$mass = \lambda \int_0^2 x^2 dx = \frac{\lambda}{3} x^3 \Big|_0^2 = \frac{8\lambda}{3}$$

$$M_x = \frac{\lambda}{2} \int_0^2 (x^2)^2 dx = \frac{\lambda}{10} x^5 \Big|_0^2 = \boxed{\frac{16\lambda}{5}}$$

$$M_y = \lambda \int_0^2 x(x^2) dx = \frac{\lambda}{4} x^4 \Big|_0^2 = \boxed{4\lambda}$$

$$\text{Centroid } (x,y) = \left(\frac{M_y}{mass}, \frac{M_x}{mass} \right) = \boxed{\left(\frac{3}{2}, \frac{6}{5} \right)}$$

5. The profile $y = \sqrt{4 - x^2}$ on the interval $x \in [-1, 1]$ is revolved around the x -axis. Find the surface area of this surface.

Use the following formula to find the surface area of an arc rotated about the x -axis:

$$S = \int 2\pi y ds$$

$$\text{where } ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

We choose to integrate with respect to x since we are given that interval.

$$S = \int_{-1}^1 2\pi y \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} dx$$

We can simplify the square root term from the ds .

$$\sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} = \sqrt{1 + \frac{x^2}{4-x^2}} = \sqrt{\frac{4-x^2}{4-x^2} + \frac{x^2}{4-x^2}} = \sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}$$

$$S = \int_{-1}^1 2\pi y \frac{2}{\sqrt{4-x^2}} dx$$

We can then write y in terms of x and simplify.

$$S = \int_{-1}^1 2\pi \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} dx$$

$$S = \int_{-1}^1 4\pi dx = \boxed{8\pi}$$

6. Compute the arc length of the function $y = 1 + 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$
- (a) $\frac{14}{9}$
(b) $\frac{10}{9}$
(c) $\frac{2}{9}\sqrt{10}$
(d) $\frac{2}{27}(10\sqrt{10} + 1)$
(e) None of the above

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$y = 1 + 2x^{\frac{3}{2}}, \frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$$S = \int_0^1 \sqrt{1 + (3x^{\frac{1}{2}})^2} dx$$

$$S = \int_0^1 \sqrt{1 + 9x} dx$$

$$u = 1 + 9x, du = 9dx, dx = \frac{1}{9}du$$

The u-bounds are: $u = 1 + 9(1) = 10$, $u = 1 + 9(0) = 1$

$$S = \sqrt{u} \frac{1}{9} du$$

$$S = \frac{1}{9} \left(\frac{2}{3}\right) u^{\frac{3}{2}} \Big|_1^{10}$$

$$\left(\frac{2}{27}\right) (10^{\frac{3}{2}} - 1^{\frac{3}{2}})$$

$$S = \left(\frac{2}{27}\right) (10\sqrt{10} - 1)$$

The answer is therefore **(e)**.

7. Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{5n^2 + 2}{\sqrt{n^4 + 7n}}$$

- (a) 0
- (b) 5
- (c) ∞
- (d) $\frac{5}{\sqrt{7}}$
- (e) None of the above

Factor out the highest power of n from the denominator.

$$a_n = \frac{5n^2 + 2}{\sqrt{n^4 + 7n}}$$

$$a_n = \frac{5n^2 + 2}{\sqrt{n^4 \left(1 + \frac{7}{n^3}\right)}}$$

$$a_n = \frac{5n^2 + 2}{n^2 \sqrt{1 + \frac{7}{n^3}}}$$

Divide numerator and denominator by n^2 :

$$a_n = \frac{5 + \frac{2}{n^2}}{\sqrt{1 + \frac{7}{n^3}}}$$

Now take the limit as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} a_n = \frac{5 + 0}{\sqrt{1 + 0}}$$

$$\lim_{n \rightarrow \infty} a_n = 5$$

The sequence therefore **converges**.

The answer is therefore **(b)**.

8. Determine whether the geometric series is convergent or divergent. If it is convergent, find the sum. (If the quantity diverges, enter DIVERGES.)

$$\sum_{n=1}^{\infty} \frac{4(-3)^{n-1}}{7^n}$$

- (a) $\frac{4}{7}$
(b) $\frac{2}{5}$
(c) $\frac{5}{2}$
(d) DIVERGES
(e) None of the above

Rewrite the series in geometric form.

$$\sum_{n=1}^{\infty} \frac{4(-3)^{n-1}}{7^n} = \sum_{n=1}^{\infty} \frac{4}{7} \left(\frac{-3}{7}\right)^{n-1}$$

This is a geometric series with

$$a_1 = \frac{4}{7}, \quad r = \frac{-3}{7}$$

Since

$$\left|\frac{-3}{7}\right| < 1,$$

the geometric series converges.

Its sum is

$$S = \frac{a_1}{1-r}$$

$$S = \frac{\frac{4}{7}}{1 - \left(\frac{-3}{7}\right)}$$

$$S = \frac{\frac{4}{7}}{1 + \frac{3}{7}}$$

$$S = \frac{\frac{4}{7}}{\frac{10}{7}}$$

$$S = \frac{4}{10} = \frac{2}{5}$$

The answer is therefore **(b)**.

9. Determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^2 + 3}{n^3 + 3}$$

$\frac{n^2+3}{n^3+3}$ can be compared to $\frac{n^2}{n^3}$ by the comparison test.

$\frac{n^2}{n^3} = \frac{1}{n}$ which diverges by the p-test ($p = 1$).

Because $\frac{n^2+3}{n^3+3} > \frac{n^2}{n^3}$, the series diverges.

10. Determine whether the series converges or diverges. Note that $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Use the divergence test: $a_n = \frac{n!}{n^n}$ and $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \left[\frac{(n+1)!}{(n+1)^{n+1}} \right] \left[\frac{n^n}{n!} \right] = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1} \right)^n$$

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{1}{n}} \right)^n = \frac{1}{e} < 1$$

So the series converges.

11. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5^n}{n + 6^n}$$

$\frac{5^n}{n+6^n} < \frac{5^n}{6^n} = \left(\frac{5}{6}\right)^n$ which converges. Thus, $\frac{5^n}{n+6^n}$ converges.

12. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{n^2}{2n^2 + 5} \right)^n$$

Use the root test:

$$\left(\frac{n^2}{2n^2 + 5} \right)^n \rightarrow [a_n]^{\frac{1}{n}} \rightarrow \left[\left(\frac{n^2}{2n^2 + 5} \right)^n \right]^{\frac{1}{n}}$$

$$= \frac{n^2}{2n^2 + 5}$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 5} = \frac{1}{2} < 1 \rightarrow \boxed{\text{Series converges}}.$$

13. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{17n^2}{3n^4 - 1}$$

Use the limit comparison test:

$$a_n = \frac{17n^2}{3n^4 - 1} \text{ and } b_n = \frac{1}{n^2}$$

$$\frac{a_n}{b_n} = \left(\frac{17n^2}{3n^4 - 1} \right) \left(\frac{n^2}{1} \right) = \frac{17}{3} \rightarrow \text{both do the same thing.}$$

$$\frac{1}{n^2} \text{ converges by the p-test} \rightarrow \boxed{\text{both converge}}.$$

14. Determine if the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(2n+1)(-2)^n}{n!}$$

$$a_n = \frac{(2n+1)(-2)^n}{n!}, \quad a_{n+1} = \frac{(2n+3)(-2)^{n+1}}{(n+1)!}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{-2(2n+3)n!}{(n+1)!(2n+1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{-2(2n+3)}{(n+1)(2n+1)} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{-4n-6}{2n^2+\dots} \right| \\ &= \lim_{n \rightarrow \infty} = 0 < 1 \end{aligned}$$

Therefore the series converges absolutely.