



## Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 241 – Calculus III

### Midterm 2 Worksheet Solutions

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*The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.*

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Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Mar. 24, 4-6 pm Noah and Rodrigo

Can't make it to a session? Here's our schedule by course:

<https://care.grainger.illinois.edu/tutoring/schedule-by-subject>

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: <https://queue.illinois.edu/q/queue/845>
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”

**Please be sure to follow the above steps to add yourself to the Queue.**

Good luck with your exam!

1. Calculate the following derivatives:

(a) Find  $\frac{df}{dt}$  for  $f(x, y) = xe^{xy}$ ,  $x(t) = t^2$ ,  $y(t) = \frac{1}{t}$

(b) Find  $f_t$  for  $f(x, y) = 2xy$ ,  $x(s, t) = st$ ,  $y(s, t) = s^2t^2$

(a) Applying the chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$\frac{\partial f}{\partial x} = e^{xy} + x(ye^{xy})$$

$$\frac{dx}{dt} = 2t$$

$$\frac{\partial f}{\partial y} = x^2e^{xy}$$

$$\frac{dy}{dt} = -\frac{1}{t^2}$$

$$\frac{df}{dt} = [e^{xy} + x(ye^{xy})](2t) + [x^2e^{xy}]\left(-\frac{1}{t^2}\right)$$

$$\boxed{\frac{df}{dt} = 2te^t + t^2e^t}$$

(b) Applying Chain Rule:

$$f_t = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = 2y, \quad \frac{\partial x}{\partial t} = s, \quad \frac{\partial f}{\partial y} = 2x, \quad \frac{\partial y}{\partial t} = 2ts^2$$

$$\boxed{f_t = 6s^3t^2}$$

2. Show that  $f(x, y) = y^2e^{xy}$  is differentiable at  $(0, 2)$  and use linear approximation to find the value of  $f(0.1, 2.1)$ .

$$f_x = y^3e^{xy}$$

$$f_y = 2ye^{xy} + xy^2e^{xy}$$

$$f_x(0, 2) = 8$$

$$f_y(0, 2) = 4$$

Both  $f_x$  and  $f_y$  are continuous functions, so  $f(x, y)$  is differentiable.

The linear approximation function of  $f(x, y)$  near  $(0, 2)$  is

$$L(x, y) = f(0, 2) + f_x(0, 2)(x - 0) + f_y(0, 2)(y - 2)$$

, so the approximated value of  $f(0.1, 2.1)$  is

$$L(0.1, 2.1) = 4 + 8(0.1) + 4(0.1) = 5.2$$

$$f(0.1, 2.1) \approx 5.2$$

3. A tiny spaceship is orbiting a path given by  $x^2 + y^2 = 4$ . The solar radiation at a point  $(x, y)$  in the plane of the orbit is  $f(x, y) = xy + 2y$ .

Use the method of *Lagrange multipliers* to find the maximum value and minimum value of solar radiation experienced by the tiny spaceship in its orbit.

The function to be maximized/minimized is  $f$ , subject to the constraint  $x^2 + y^2 = 4$ , which we will call  $g$ .

$$\nabla f = \langle y, x + 2 \rangle \quad \nabla g = \langle 2x, 2y \rangle$$

We have 3 equations to work with. We know

$$\nabla f = \lambda \nabla g$$

gives the following system of equations

$$\begin{aligned} y &= \lambda(2x) \\ x + 2 &= \lambda(2y) \\ x^2 + y^2 &= 4 \end{aligned}$$

Where the last equation is the constraint. We solve for  $\lambda$  in the first two equations:

$$\frac{y}{2x} = \lambda \text{ and } \frac{x + 2}{2y} = \lambda$$

Set them equal to each other to get an equation with just  $x$  and  $y$ :

$$\begin{aligned} \frac{y}{2x} &= \frac{x + 2}{2y} \\ 2y^2 &= 2x^2 + 4x \\ y^2 &= x^2 + 2x \end{aligned}$$

Plug this in to the constraint

$$\begin{aligned} x^2 + y^2 &= 4 \\ x^2 + x^2 + 2x &= 4 \end{aligned}$$

We get  $x = 1, -2$

Obtain the corresponding values for  $y$ :

For  $x = 1$

$$y^2 = x^2 + 2x \quad y = -\sqrt{3}, \sqrt{3}$$

For  $x = -2$

$$y^2 = x^2 + 2x \quad y = 0$$

Now we must test whether these are minimum or maximum points. We have three points to test in our radiation function  $f(x, y) = xy + 2y$

$$f(1, \sqrt{3}) = 3\sqrt{3}$$

$$f(1, -\sqrt{3}) = -3\sqrt{3}$$

$$f(-2, 0) = 0$$

$$\boxed{\text{Max: } 3\sqrt{3}}$$

$$\boxed{\text{Min: } -3\sqrt{3}}$$

4. Let  $f(x, y)$  be a differentiable function on the disk  $\{D : x^2 + y^2 \leq 400\}$ , where:

- (I)  $f(x, y) = 19$  for every point on the boundary of the disk  $x^2 + y^2 = 400$
- (II)  $f(0, 0) = 7$
- (III)  $f(x, y)$  has only one critical point which is at  $(-1, 2)$

Decide which statement is true:

- A)  $f(-1, 2) > 7$
- B)  $f(-1, 2) < 7$
- C)  $f(-1, 2) = 7$
- D) Not enough information is given

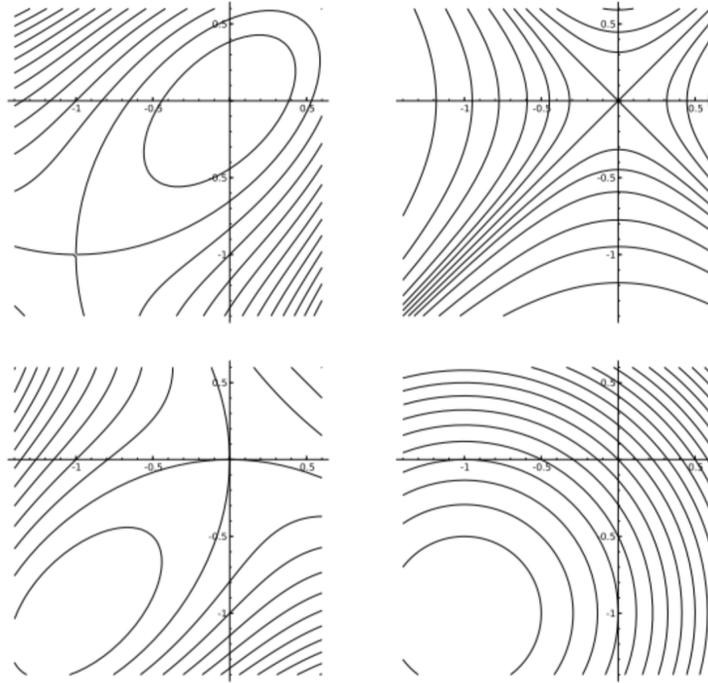
From the extreme value theorem we know that the maximum and minimum of the function in the domain  $D$  must (if they exist) be at the critical points or some points on the boundary.

Suppose  $f(-1, 2) \geq 7$ , then the minimum of the function is not at the point  $(-1, 2)$  since  $f(0, 0)$  has a smaller value. This would imply another critical point below  $f(-1, 2)$ , but the function only has one critical point.

The minimum is also not on the boundary since  $f(x, y) = 19 > 7$  for every point on the boundary of the disk.

Lastly, suppose  $f(-1, 2) = 7$ , then this implies that  $(0, 0)$  is also a critical point, which disagrees with the condition. So the only way to satisfy the condition that there is one critical point is make  $f(-1, 2) < 7$ . This means it is below the origin.

5. Consider the function  $f(x, y) = x^3 + y^3 + 3xy$
- (a) The critical points of  $f$  are  $(0, 0)$  and  $(-1, -1)$ . Classify them into local minima, local maxima and/or saddle points
- (b) Based on your answer in (a), identify the correct contour diagram of  $f$



$$(a) \quad f_x = 3x^2 + 3y$$

$$f_{xx} = 6x$$

$$f_y = 3y^2 + 3x$$

$$f_{yy} = 6y$$

$$f_{xy} = f_{yx} = 3$$

$$\text{At } (0, 0) \quad D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = -9 < 0 \rightarrow \text{SADDLE}$$

$$\text{At } (-1, -1) \quad D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} = 36 - 9 = 27$$

$$27 > 0 \text{ and } f_{xx} = -6 < 0 \rightarrow \text{LOCAL MAX}$$

- (b) Bottom left is correct (max/min values have no level curves within it, and saddle points have cross-over shapes)

6. What is the partial derivative of  $f(x, y, z) = e^x \sin(yz)z^3 \ln(y)$  with respect to  $x$ .  
It's the same function.

7. Consider a function  $f(t) = f(x(t), y(t), z(t)) = xyz - z^2$ , where  $x(t)$ ,  $y(t)$ ,  $z(t)$  are defined as followed:

$$x(t) = 2t^2 + 1$$

$$y(t) = 3 - \frac{1}{t}$$

$$z(t) = 3$$

Find the following values:

(a)  $f_z(3, 1, 2)$

(b)  $\left. \frac{dx}{dt} \right|_{(t=0)}$

(c)  $\left. \frac{df}{dt} \right|_{(t=1)}$

(a)

$$\frac{\partial}{\partial z}(xyz - z^2)$$

$$xy - 2z$$

$$f_z(3, 1, 2) = 3 - 4 = -1$$

(b)

$$\frac{d}{dt}x = 4t$$

$$\left. \frac{dx}{dt} \right|_{(t=0)} = 0$$

(c)

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = yz(4t) + xz\left(\frac{1}{t^2}\right) + (xy - 2z)(0)$$

When  $t = 1$ :

$$x = 3$$

$$y = 2$$

$$z = 3$$

Plug into the equation:

$$\left. \frac{df}{dt} \right|_{(t=1)} = 24 + 9 + 0 = 33$$

8. Consider the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4}$$

Does this limit exist? If so, what is its value? Justify your answer.

Along  $x = 0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4} = \lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{y^4} = 1$$

Along  $x = y$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4} = \lim_{(x,y) \rightarrow (0,0)} \frac{\cos^2 x}{2} = \frac{1}{2}$$

Since  $0 \neq 1/2$ , the Limit DNE

9. If  $f(x, y, z) = xye^z$ , find the gradient of  $f$  and the directional derivative at  $(2, 5, 0)$  in the direction of  $\vec{v} = 2\hat{i} - \hat{j} + \hat{k}$ .

$$f_x = ye^z$$

$$f_y = xe^z$$

$$f_z = xye^z$$

The gradient of  $f$  is  $\nabla f(x, y, z) = \langle ye^z, xe^z, xye^z \rangle$

The unit vector of  $\vec{v}$  is

$$\vec{u} = \left\langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

At  $(2, 5, 0)$ ,

$$\nabla f(2, 5, 0) = \langle 5, 2, 10 \rangle$$

The directional derivative at  $(2, 5, 0)$  in the direction of  $\vec{v}$  is calculated to be

$$D_{\vec{u}} f(2, 5, 0) = \nabla f(2, 5, 0) \cdot \vec{u} = \langle 5, 2, 10 \rangle \cdot \left\langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$= \frac{18}{\sqrt{6}} = \boxed{3\sqrt{6}}$$

10. Evaluate the following functions with  $\lim_{(x,y) \rightarrow (0,0)}$ :

$$(a) f(x, y) = \frac{3xy - x^2y}{x^2 + y^2 + xy}$$

$$(b) f(x, y) = \frac{y \sin(x) + y^2 e^x}{y}$$

$$(c) f(x, y) = \frac{(x^2 + y^2)^5}{x^{10} + y^4}$$

(a) Use polar coordinates:

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{r \rightarrow 0} f(r, \theta)$$

$$\lim_{r \rightarrow 0} \frac{3r^2 \sin \theta \cos \theta - r^3 \cos^2 \theta \sin \theta}{r^2 (\cos^2 \theta + \sin^2 \theta + \cos \theta \sin \theta)}$$

$$\lim_{r \rightarrow 0} \frac{3 \sin \theta \cos \theta - r \cos^2 \theta \sin \theta}{(1 + \cos \theta \sin \theta)}$$

Let  $r \rightarrow 0$  (eliminating the  $\cos(\theta)^2 \sin(\theta)$  term). We end up with

$$\frac{3 \sin \theta \cos \theta}{(1 + \cos \theta \sin \theta)}$$

Plug in  $\theta = 0$  and  $\theta = \frac{\pi}{4}$

$$\begin{aligned} \theta &= 0 \\ \frac{0}{1} &= 0 \end{aligned}$$

$$\begin{aligned} \theta &= \frac{\pi}{4} \\ \frac{\frac{3}{2}}{1 + \frac{1}{2}} &= 1 \end{aligned}$$

The limit has different values for different  $\theta$ , thus the limit DOES NOT EXIST

(b) Divide out a  $y$ , and the let  $(x, y) \rightarrow (0, 0)$

$$\lim_{(x,y) \rightarrow (0,0)} \sin(x) + ye^x \rightarrow \boxed{0}$$

(c) Take the path  $x = 0$  and you end up with  $f(0, y) = y^6$ , which has a limit of 0  
 Take the path  $y = 0$  and you end up with  $f(x, 0) = \frac{x^{10}}{x^{10}}$  which has a limit of 1

Thus, the limit DOES NOT EXIST

11. Find min/max of  $f(x,y,z) = 3x^2 + 8y^2 + z^2 - 2z$  defined on the domain  $x^2 + 4y^2 + 2z \leq 8$  and  $z \geq 0$
- (a) The domain is (select all that apply)
- I) open
  - II) closed
  - III) bounded
  - IV) unbounded
- (b) Where are the critical points inside the domain? Evaluate the function value on these points.
- (c) What is the minimum and maximum on  $x^2 + 4y^2 + 2z = 8$ ?
- (d) What is the minimum and maximum on  $z = 0$ ?
- (e) What is the global minimum and maximum of the whole domain?

- (a) The region includes *all* of its boundary points, therefore it's closed and bounded (II and III).
- (b) The critical points inside the domain are where  $\nabla f = 0$ .

$$\nabla f = \langle 6x, 16y, 2z - 2 \rangle = \langle 0, 0, 0 \rangle$$

Which gives us

$$x = 0, y = 0, z = 1$$

Critical Points:  $f(0, 0, 1) = -1$

- (c) Since we're dealing with the boundary now, we must use Lagrange multipliers. Start by defining  $g(x, y, z)$  as the boundary of the curve with  $g(x, y, z) = 8$ .

$$g(x, y, z) = x^2 + 4y^2 + 2z$$

$$\nabla f = \lambda \nabla g \rightarrow \langle 6x, 16y, 2z - 2 \rangle = \langle 2\lambda x, 8\lambda y, 2\lambda \rangle$$

Move  $\nabla f$  and  $\lambda \nabla g$  to the same side to solve for each variable.

$$\begin{aligned} 6x - 2\lambda x &= 0 \\ \lambda = 3 \text{ or } x &= 0 \end{aligned}$$

$$\begin{aligned} 16y - 8\lambda y &= 0 \\ \lambda = 2 \text{ or } y &= 0 \end{aligned}$$

$$\begin{aligned} 2z - 2 - 2\lambda &= 0 \\ \lambda = z - 1 \end{aligned}$$

$$g(x, y, z) = x^2 + 4y^2 + 3z = 12$$

Solving these four equations for four unknowns, we have three solutions:

$$\lambda = 3, x = 0, y = 0, z = 4$$

$$\lambda = 2, x = 0, y = \pm \frac{1}{\sqrt{2}}, z = 3$$

$f(0, 0, 4) = 8 \quad f\left(0, \pm \frac{1}{\sqrt{2}}, 3\right) = 7$

(d) We are using a new boundary now, so we have to define a new  $g(x, y, z)$ .

$$g(x, y, z) = z$$

$$\nabla f = \lambda \nabla g \rightarrow \langle 6x, 16y, 2z - 2 \rangle = \langle 0, 0, \lambda \rangle$$

$$g(x, y, z) = 0$$

There is one solution  $x = 0, y = 0, z = 0, \lambda = -2$

$$f(0, 0, 0) = 0$$

(e)

$$f(0, 0, 1) = -1 \quad f(0, 0, 4) = 8$$

12. Compute the double integral over the indicated rectangle. Confirm your answer by switching the order of integration and recomputing.

$$\iint_R 2x - 4y^3 \, dA \quad R = [-5, 4] \times [0, 3]$$

$$\int_0^3 \int_{-5}^4 2x - 4y^3 \, dx \, dy$$

$$\int_0^3 -9 - 36y^3 \, dy$$

$$-9y - 9y^4 \Big|_0^3 = \boxed{-756}$$

$$\int_{-5}^4 \int_0^3 2x - 4y^3 \, dy \, dx$$

$$\int_{-5}^4 6x - 81 \, dx$$

$$3x^2 - 81x \Big|_{-5}^4 = \boxed{-756}$$

13. Making an appropriate change of variables, compute the following double integral over the region bound by a circle of radius 2 and a circle of radius 5.

$$\iint_D e^{x^2+y^2} dA$$

Using polar coordinates gives the following integral

$$\iint_D e^{x^2+y^2} dA = \int_0^{2\pi} \int_2^5 r e^{r^2} dr d\theta$$

Then using a  $u$ -substitution  $u = r^2$  for the  $r$  dependence

$$\frac{1}{2} \int_0^{2\pi} \int_4^{25} e^u du d\theta = \boxed{\pi(e^{25} - e^4) \approx 2.26 \times 10^{11}}$$