The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 2 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Partial Derivatives

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$f(x,y) \quad \Rightarrow \quad f_x(x,y) = \frac{\partial f}{\partial x} \ \& \ f_y(x,y) = \frac{\partial f}{\partial y}$$

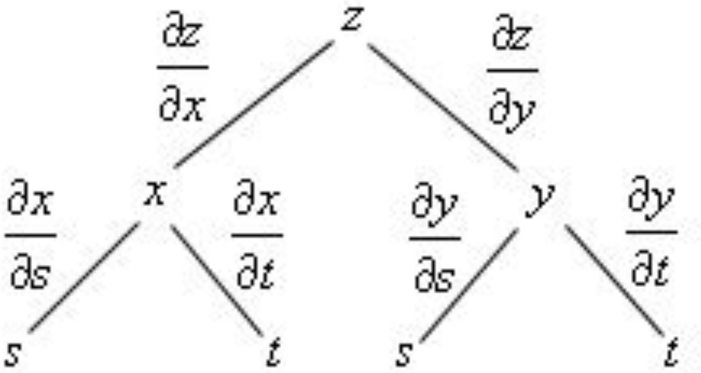
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

Chain Rule



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

change in z with respect to t

z_x

change in x with respect to t

Linear Approximation & Tangent planes

- If z = f(x, y) and f is **differentiable** at (a, b), then the value of f(m, n) can be approximated by

$$f(\mathrm{m}, \mathrm{n}) \approx L(\mathrm{m}, \mathrm{n})$$

$$L(\mathrm{m}, \mathrm{n}) = \mathrm{f}(\mathrm{a}, \mathrm{b}) + \mathrm{f}_{\mathrm{x}}(\mathrm{a}, \mathrm{b}) \cdot (\mathrm{m} - \mathrm{a}) + \mathrm{f}_{\mathrm{y}}(\mathrm{a}, \mathrm{b}) \cdot (\mathrm{n} - \mathrm{b})$$

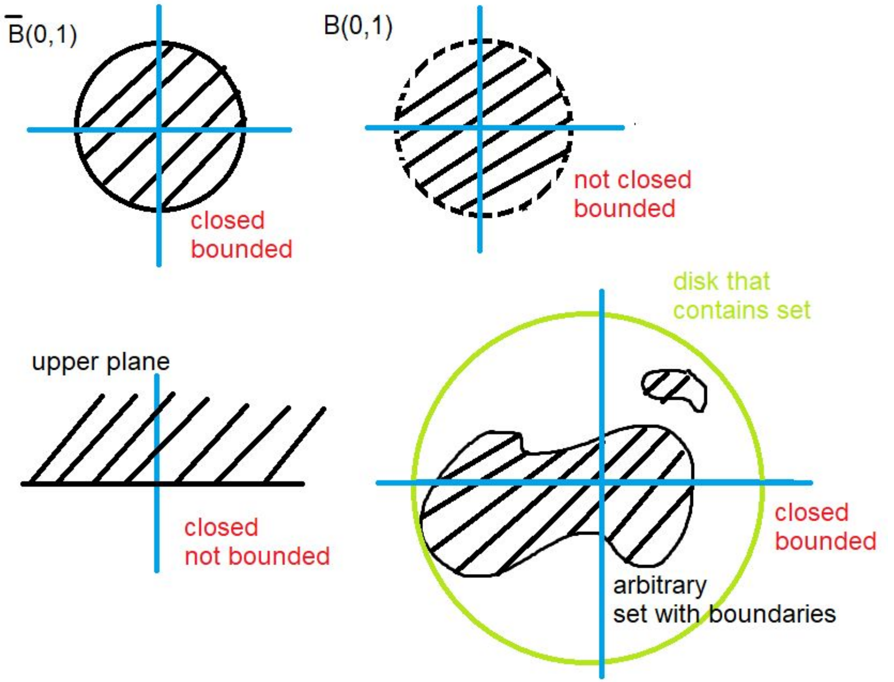
Limits and Continuity

- When computing multivariable limits,
  - Check **multiple paths** (lines and power functions) to see if there are conflicting values. If so, limits DNE
  - **Factor** (difference of squares)
  - Use **polar coordinates**
  - Try **squeeze theorem**

Extreme Value Theorem

If f(x,y) is continuous on a closed and bounded set D, then it is guaranteed that f has an absolute minimum and maximum value

- The absolute min and max will either occur at the critical points of f, or on the endpoints of the boundary D



https://math.stackexchange.com/questions/1190640/what-is-the-difference-between-closed-and-bounded-in-terms-of-domains

Gradient and Directional Derivatives

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The gradient will always point **perpendicular to the level curves/surfaces of f**
- $\nabla f = 0$ at a local minimum/maximum

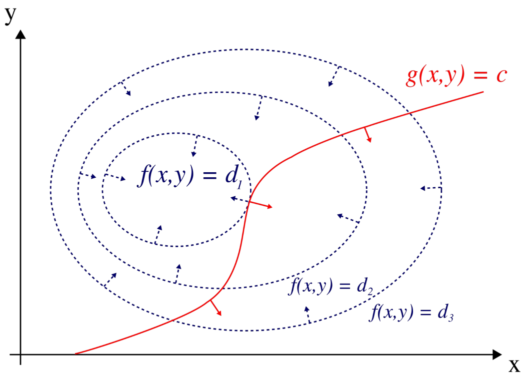
$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

- Tells you how the function f changes along the vector u

Lagrange Multiplier

- Solve the following system of equations for λ (Lagrange Multiplier)
  - Where f is the function, and g is the constraint

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
$$g(x, y, z) = k$$



Double Integrals

$$V = \iint_R f(x,y)\, dA.$$

$$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy = \iint_R f(x,y)\, dy\, dx.$$

Center of Mass

- The x, y coordinates of the center of mass for an object that has a density function ρ(x,y)

$$\bar{\mathrm{x}} = \frac{1}{\mathrm{m}} \iint x \cdot \rho(x, y) dA \qquad \bar{\mathrm{y}} = \frac{1}{\mathrm{m}} \iint y \cdot \rho(x, y) dA$$

, where mass is calculated as $\mathrm{m} = \iint \rho(x, y) dA$

Polar Coordinates

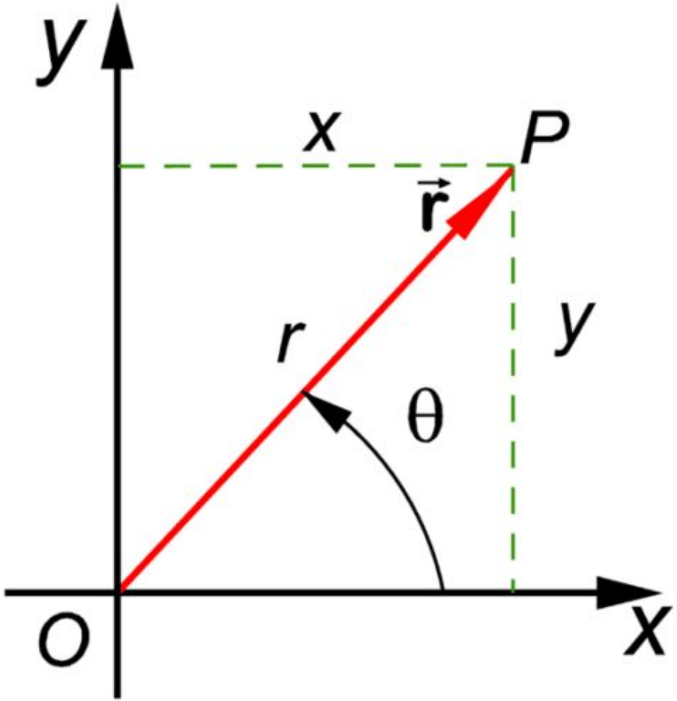
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

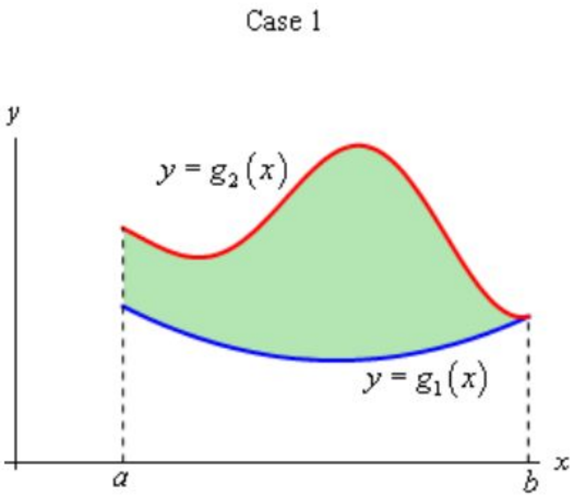
$$\theta = arctan\left(\frac{y}{x}\right)$$

$$dA = rdrd\theta$$

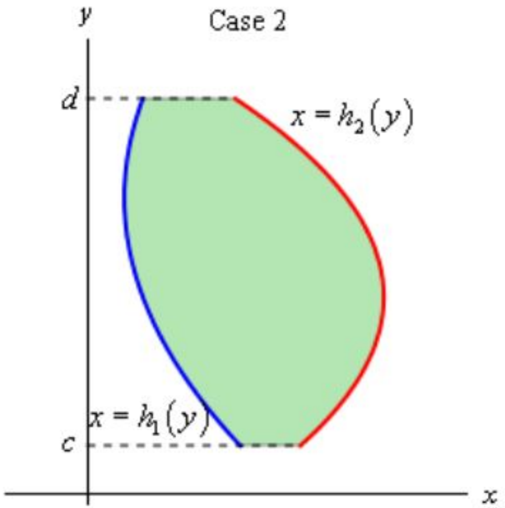


https://magoosh.com/hs/ap-calculus/2017/ap-calculus-bc-review-polar-functions/

Double Integral Over General Regions



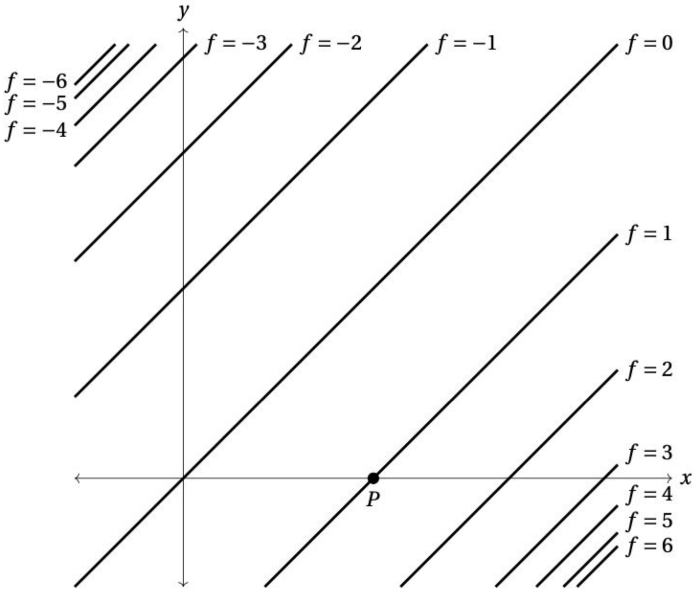
$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

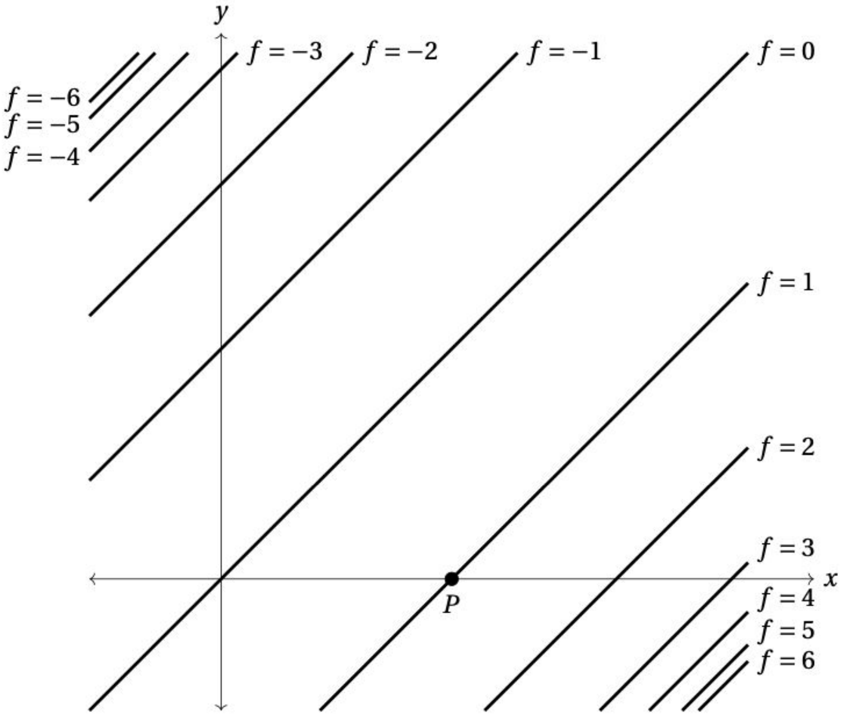
Example Question #1

- A contour map for a function f of x, y, and a point P in the plane are given below. Determine if the following quantities are negative, zero, or positive: $f_x(P)$, $f_{xx}(P)$, $f_{xy}(P)$



Example Solution #1

- $f_x(P)$: positive
- $f_{xx}(P)$: positive
- $f_{xy}(P)$: negative



Example Question #2

- Calculate the following derivative:

Find f_t for $f(x,y) = 2xy$, $x(s,t) = st$, $y(s,t) = s^2t^2$

Example Solution #2

$$\text{Find } f_t \text{ for } f(x,y) = 2xy,\ x(s,t) = st,\ y(s,t) = s^2t^2$$

Applying Chain Rule:

$$f_t = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = 2y,\ \frac{\partial x}{\partial t} = s,\ \frac{\partial f}{\partial y} = 2x,\ \frac{\partial y}{\partial t} = 2ts^2$$

$$\boxed{f_t = 6s^3t^2}$$

Example Question #3

- Consider the function $f = x^3 + y^3 + 3xy$. If the critical points of f are (0, 0) and (–1, –1), classify them into local mins, maxes, and saddle points.

Example Solution #3

- Consider the function f = $x^3 + y^3 + 3xy$. If the critical points of f are (0, 0) and (–1, –1), classify them into local mins, maxes, and saddle points.

$$f_x = 3x^2 + 3y \quad f_y = 3y^2 + 3x \quad f_{xx} = 6x \quad f_{yy} = 6y$$

$$f_{xy} = f_{yx} = 3$$

$$\text{At } (0, 0), D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 \rightarrow \text{Saddle Point}$$

$$\text{At } (-1, -1), D = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 \rightarrow \text{Because } f_{xx} = -6 < 0 \rightarrow \text{Local Max}$$