I ILLINOIS

Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 241 – Calculus III

CBTF Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Mar. 23, 6-8pm Meredith, Gabe

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/845
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Compute the double integral over the indicated rectangle. Confirm your answer by switching the order of integration and recomputing.

$$\iint_R 2x - 4y^3 \; dA \qquad R = [-5, 4] \times [0, 3]$$

$$\int_0^3 \int_{-5}^4 2x - 4y^3 \; dxdy$$

$$\int_0^3 -9 - 36y^3 \; dy$$

$$-9y - 9y^4 \Big|_0^3 = \boxed{-756}$$

$$\int_{-5}^4 \int_0^3 2x - 4y^3 \; dydx$$

$$\int_{-5}^4 6x - 81 \; dx$$

$$3x^2 - 81x \Big|_{-5}^4 = \boxed{-756}$$

2. Calculate the following derivatives:

(a) Find $\frac{\mathrm{d}f}{\mathrm{d}t}$ for $f(x, y) = xe^{xy}$, $x(t) = t^2$, $y(t) = \frac{1}{t}$

(b) Find f_t for $f(x, y) = 2xy$, $x(s, t) = st$, $y(s, t) = s^2t^2$

(a) Applying the chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\partial f}{\partial x} = e^{xy} + x(ye^{xy}) \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial f}{\partial y} = x^2e^{xy} \qquad \frac{dy}{dt} = -\frac{1}{t^2}$$

$$\frac{df}{dt} = \left[e^{xy} + x(ye^{xy})\right](2t) + \left[x^2e^{xy}\right]\left(-\frac{1}{t^2}\right)$$

$$\boxed{\frac{df}{dt} = 2te^t + t^2e^t}$$

(b) Applying Chain Rule:

$$f_t = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = 2y, \; \frac{\partial x}{\partial t} = s, \; \frac{\partial f}{\partial y} = 2x, \; \frac{\partial y}{\partial t} = 2ts^2$$

$$\boxed{f_t = 6s^3t^2}$$

3. Let $f(x, y)$ be a differentiable function on the disk $\{D : x^2 + y^2 \leq 400\}$, where:

(I) $f(x, y) = 19$ for every point on the boundary of the disk $x^2 + y^2 = 400$

(II) $f(0, 0) = 7$

(III) $f(x, y)$ has only one critical point which is at $(-1, 2)$

Decide which statement is true:

A) $f(-1, 2) > 7$

B) $f(-1, 2) < 7$

C) $f(-1, 2) = 7$

D) Not enough information is given

From the extreme value theorem we know that the maximum and minimum of the function in the domain D must (if they exist) be at the critical points or some points on the boundary.

Suppose $f(-1, 2) \geq 7$, then the minimum of the function is not at the point $(-1, 2)$ since $f(0, 0)$ has a smaller value. This would imply another critical point below $f(-1, 2)$, but the function only has one critical point.

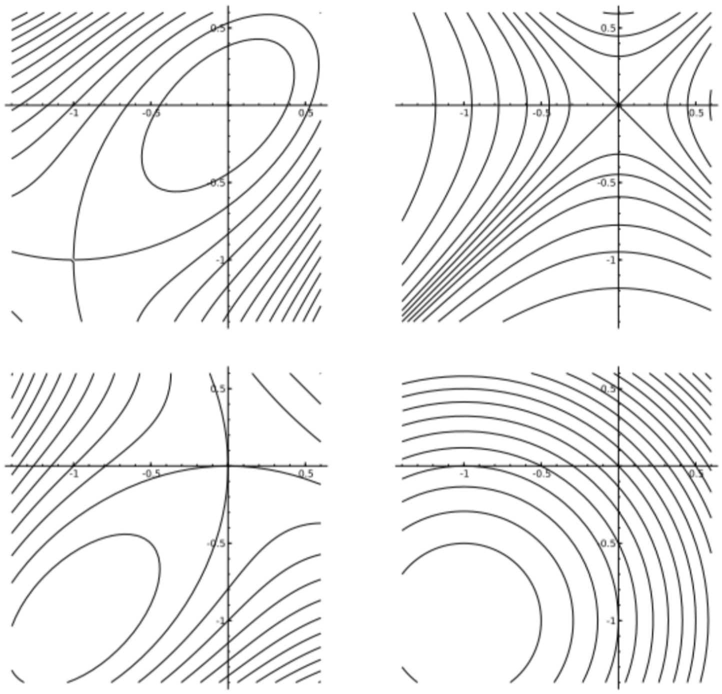
The minimum is also not on the boundary since $f(x, y) = 19 > 7$ for every point on the boundary of the disk.

Lastly, suppose $f(-1, 2) = 7$, then this implies that $(0, 0)$ is also a critical point, which disagrees with the condition. So the only way to satisfy the condition that there is one critical point is make $f(-1, 2) < 7$. This means it is below the origin.

4. Consider the function $f(x, y) = x^3 + y^3 + 3xy$

(a) The critical points of f are $(0, 0)$ and $(-1, -1)$. Classify them into local minima, local maxima and/or saddle points

(b) Based on your answer in (a), identify the correct contour diagram of f



(a) $f_x = 3x^2 + 3y$ $f_y = 3y^2 + 3x$

$f_{xx} = 6x$ $f_{yy} = 6y$

$$f_{xy} = f_{yx} = 3$$

$$\text{At } (0, 0)\ D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = -9 < 0 \to \text{SADDLE}$$

$$\text{At } (-1, -1)\ D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} = 36 - 9 = 27$$

$$27 > 0 \text{ and } f_{xx} = -6 < 0 \to \text{LOCAL MAX}$$

(b) Bottom left is correct (max/min values have no level curves within it, and saddle points have cross-over shapes)

5. Find min/max of f(x,y,z) = $3x^2 + 8y^2 + z^2 - 2z$ defined on the domain $x^2 + 4y^2 + 2z \leq 8$ and z ≥ 0

(a) The domain is (select all that apply)

I) open

II) closed

III) bounded

IV) unbounded

(b) Where are the critical points inside the domain? Evaluate the function value on these points.

(c) What is the minimum and maximum on $x^2 + 4y^2 + 2z = 8$?

(d) What is the minimum and maximum on z = 0?

(e) What is the global minimum and maximum of the whole domain?

(a) The region includes *all* of its boundary points, therefore it's closed and bounded (II and III).

(b) The critical points inside the domain are where $\nabla f = 0$.

$$\nabla f = \langle 6x, 16y, 2z - 2 \rangle = \langle 0, 0, 0 \rangle$$

Which gives us

$$x = 0,\ y = 0,\ z = 1$$

$$\boxed{\text{Critical Points: } f(0,0,1) = -1}$$

(c) Since we're dealing with the boundary now, we must use Lagrange multipliers. Start by defining $g(x, y, z)$ as the boundary of the curve with $g(x, y, z) = 8$.

$$g(x, y, z) = x^2 + 4y^2 + 2z$$

$$\nabla f = \lambda \nabla g \to \langle 6x, 16y, 2z - 2 \rangle = \langle 2\lambda x, 8\lambda y, 2\lambda \rangle$$

Move ∇f and $\lambda \nabla g$ to the same side to solve for each variable.

$6x - 2\lambda x = 0$
$\lambda = 3$ or $x = 0$

$16y - 8\lambda y = 0$
$\lambda = 2$ or $y = 0$

$2z - 2 - 2\lambda = 0$
$\lambda = z - 1$

$$g(x, y, z) = x^2 + 4y^2 + 3z = 12$$

Solving these four equations for four unknowns, we have three solutions:

$\lambda = 3,\ x = 0,\ y = 0,\ z = 4$

$\lambda = 2,\ x = 0,\ y = \pm\frac{1}{\sqrt{2}},\ z = 3$

$$\boxed{f(0,0,4) = 8 \qquad f(0, \pm\frac{1}{\sqrt{2}}, 3) = 7}$$

(d) We are using a new boundary now, so we have to define a new $g(x, y, z)$.

$$g(x, y, x) = z$$

$$\nabla f = \lambda \nabla g \to \langle 6x, 16y, 2z - 2\rangle = \langle 0, 0, \lambda\rangle$$

$$g(x, y, z) = 0$$

There is one solution $x = 0,\ y = 0,\ z = 0,\ \lambda = -2$

$$\boxed{f(0, 0, 0) = 0}$$

(e)

$$\boxed{f(0, 0, 1) = -1 \qquad f(0, 0, 4) = 8}$$

6. If $f(x,y,z) = xye^z$, find the gradient of f and the directional derivative at (2,5,0) in the direction of $\vec{v} = 2\hat{i} - \hat{j} + \hat{k}$.

$f_x = ye^z$ $\quad$ $f_y = xe^z$ $\quad$ $f_z = xye^z$

The gradient of f is $\boxed{\nabla f(x,y,z) = \langle ye^z, xe^z, xye^z\rangle}$

The unit vector of $\vec{v}$ is

$$\vec{u} = \langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle$$

At (2,5,0),

$$\nabla f(2,5,0) = \langle 5,2,10\rangle$$

The directional derivative at (2,5,0) in the direction of $\vec{v}$ is calculated to be

$$D_u f(2,5,0) = \nabla f(2,5,0)\cdot\vec{u} = \langle 5,2,10\rangle\cdot\langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle$$

$$= \frac{18}{\sqrt{6}} = \boxed{3\sqrt{6}}$$

7. Compute the double integral over the indicated rectangle. Confirm your answer by switching the order of integration and recomputing.

$$\iint_R 2x - 4y^3\ dA \qquad R = [-5,4]\times[0,3]$$

$$\int_0^3\int_{-5}^4 2x - 4y^3\ dxdy$$

$$\int_0^3 -9 - 36y^3\ dy$$

$$-9y - 9y^4\Big|_0^3 = \boxed{-756}$$

$$\int_{-5}^4\int_0^3 2x - 4y^3\ dydx$$

$$\int_{-5}^4 6x - 81\ dx$$

$$3x^2 - 81x\Big|_{-5}^4 = \boxed{-756}$$