MATH 257 Exam 2 CARE Review

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In-Person Resources

CARE Drop-in tutoring:
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Course Office hours:
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Focused on lectures and labs
CAs: On Zoom, schedule on Canvas.
Focused on Python
Instructors:
Chuang: Mon 1-2 PM, 233 CAB
Leditzky: Mon 4:15-5:15 PM, 204B Harker Hall

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 257	12pm-6pm	12pm-2pm 4pm-7pm 8pm-10pm	1pm-3pm 5pm-10pm	12pm-7pm	2pm-9pm	12pm-2pm 3pm-5pm	

Topic Summary

- LU Decomposition
    - Lower/Upper Triangular Matrix
    - LU for Linear Systems
    - Permutation Matrix
- Orthogonal Matrices
    - Inner Product
    - Orthogonality
- Subspaces
    - Column Space
    - Null Space
    - Linear Independence
    - Fundamental Subspaces
- Basis and Dimension
    - Fundamental Subspaces
    - Orthonormal bases
- Graph and Adjacency Matrices
- Coordinates
    - Coordinate Matrices
    - Orthogonal/normal Complements

Upper/Lower Triangular Matrices

Upper Triangular:

$$\begin{bmatrix} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Finding this is like doing REF with only row replacement

Lower Triangular:

$$\begin{bmatrix} \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \ddots & \vdots \\ \star & \star & \star & \star & \star \end{bmatrix}$$

Keep track of your row operations to find L

LU Decomposition:

A = LU

- Not all matrices have LU decompositions
- LU decompositions are not unique (unlike inverses)

Finding the LU Decomposition

Determine the LU-decomposition of $\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$

1) Col 1 Row 2 2) Col 1 Row 3 3) Col 3 Row 3

$$\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix} \underset{R_2 \to R_2 \boxed{-4} R_1, R_3 \to R_3 \boxed{-4} R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & -4 & 0 \end{bmatrix} \underset{R_3 \to R_3 \boxed{-1} R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$, L := \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \qquad U := \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

LU for Linear Systems

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}}_{\mathbf{b}}$$

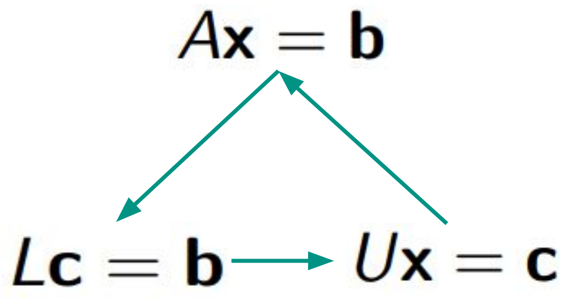
$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}}_{\mathbf{c}}.$$

Use **LU decomposition** to solve a linear system if:

1. A is nxn matrix
2. A = LU
3. b ∈ R^n

Step-by-step Algorithm

1. Find L and U
2. Solve for c using Lc = b
3. Solve for x using Ux = c



$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = L\mathbf{c} = \mathbf{b}$$

You Try: Solving with LU Decomposition

$$Given: A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$Solve \;\; A\vec{x} = \vec{b} \quad where \;\; b = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

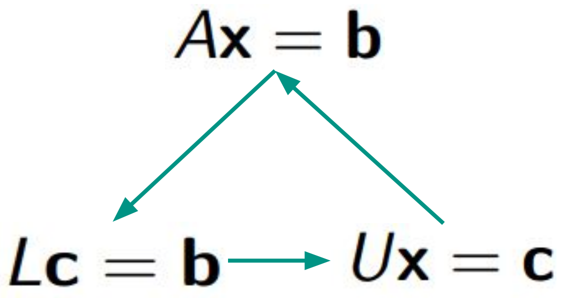
Do not compute A

provide values of $\vec{x}$ and the intermediate vector $\vec{c}$

For your reference:

Step-by-step Algorithm

1. Find L and U
2. Solve for c using Lc = b
3. Solve for x using Ux = c



$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = L\mathbf{c} = \mathbf{b}$$

Solution: Solving with LU Decomposition

Begin $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$I \quad Substitute\ U\vec{x} = \vec{c}$

$II \quad Solve\ L\vec{c} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \xrightarrow{\text{fwd. sub.}} \vec{c} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$III \quad Solve\ U\vec{x} = \vec{c}$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \xrightarrow{\text{back sub.}} \vec{x} = \begin{bmatrix} -1 \\ -6 \\ -2 \end{bmatrix}$$

Permutation Matrices: for matrices that don't have an LU decomposition

Theorem 21. *Let A be $n \times n$ matrix. Then there is a permutation matrix P such that PA has an LU-decomposition.*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = PA$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Step-by-step:

- Use the interchange operation done on **A** to an equivalent size identity matrix, this will be your **P** matrix
- Solve the for the **LU** decomposition of **PA**

When we apply the $\mathbf{P}^{-1}$ to **LU** (on the right), we'll be able to get the original value of **A**

$$PA = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:U}$$

Inner Product, Norm, and Distance

If $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w}$ is

$$v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

The **inner product** of $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

AKA the dot product
It is a scalar!

Definition. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

The **norm** (or **length**) of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

The **distance** between $\mathbf{v}$ and $\mathbf{w}$ is

$$\text{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|.$$

The norm is also a scalar!

Properties of the Inner Product: similar to scalars

Theorem 22. *Let* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$, *and let* c *be any scalar. Then*

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative!

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ Distributive!

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ Associative!

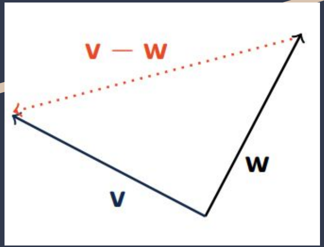
(d) $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$, *and* $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ *if and only if* $\mathbf{u} = \mathbf{0}$.

Orthogonality

(fancy word for perpendicular)

Vectors are orthogonal if their **dot product is zero**.

Why? The dot product of two non-zero vectors can only be zero if the angle between them is 90.



Orthonormality

A **unit vector** in $\mathbb{R}^n$ is vector of length 1.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

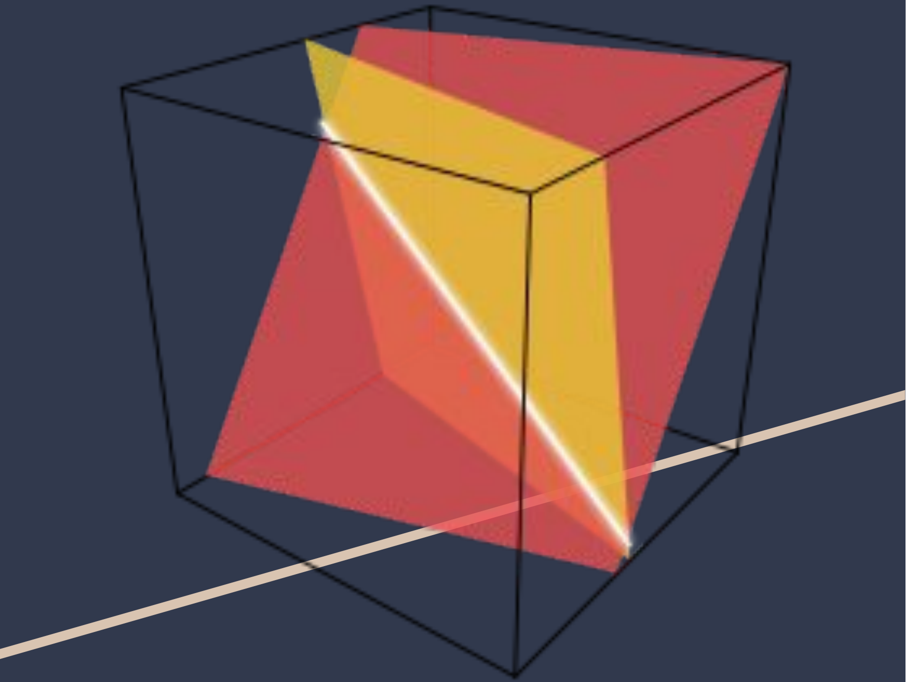
Orthonormal sets are all orthogonal to each other and unit vectors.

Ex.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Subspaces



W is a **subspace** of **V,** if:

- W contains the 0 vector
- Adding any 2 vectors in W together gives a vector also in W
- Multiplying any vector in W by any scalar gives a vector also in W

Theorem 24. *Let* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in \mathbb{R}^n$. *Then Span* $(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$ *is a subspace of* $\mathbb{R}^n$.

Vector Spaces ‘V’: a specific type of subspace

$u, v, w \in V$ and for all scalars $c, d \in \mathbb{R}$:

- $\mathbf{u} + \mathbf{v}$ is in V. (V is “closed under addition”.)
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- There is a vector (called the zero vector) $\mathbf{0}_V$ in V such that $\mathbf{u} + \mathbf{0}_V = \mathbf{u}$.
- For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}_V$.
- $c\mathbf{u}$ is in V. (V is “closed under scalar multiplication”.)
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- $(cd)\mathbf{u} = c(d\mathbf{u})$.
- $1\mathbf{u} = \mathbf{u}$.

Column Spaces

Definition. The **column space**, written as Col(A), of an $m \times n$ matrix A is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix}$, then
$\text{Col}(A) = \text{span}(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$.

$$A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} \underline{1} & -10 & -24 & -42 \\ 0 & \underline{2} & 6 & 10 \\ 0 & 0 & \underline{3} & 3 \end{bmatrix}$$

$$\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix}, \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \right\}$$

How to solve for Col(A):

1. Put matrix A into REF
2. Find all the pivots of A
3. Map the pivots to the columns of your original matrix, A

Null Spaces

Definition. The **nullspace** of an $m \times n$ matrix A, written as $\text{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$; that is, $\text{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^n \ : \ A\mathbf{v} = \mathbf{0}\}$.

How to solve for Nul(A):

1. Set matrix A into **Augmented Matrix** with zeros on the right (**$A\mathbf{x} = \mathbf{0}$**)
2. Get A into **RREF**
3. Solve for **x**

$\text{Nul}(A) = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \ldots)$

Null Space Example

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

RREF

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $x_1 - 2x_2 - x_4 + 3x_5 = 0$
- $x_3 + 2x_4 - 2x_5 = 0$

$$x_1 = 2x_2 + x_4 - 3x_5$$
$$x_3 = -2x_4 + 2x_5$$

$$\begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\text{Nul}\, A = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Linear Independence

Definition. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).
We say the vectors are **linearly dependent** if they are not linearly independent.

Theorem 30. *Let A be an $m \times n$ matrix. The following are equivalent:*

- *The columns of A are linearly independent.*
- *$A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.*
- *A has n pivots.*
- *there are no free variables for $A\mathbf{x} = \mathbf{0}$.*

Basis and Dimension

Definition. Let V be a vector space. A sequence of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ in V is a **basis** of V if

- ➔ $V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$, and
- ➔ $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ are linearly independent.

The number of vectors in a basis of V is the **dimension** of V.

Basis and Dimension example

Is $\left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix}\right)$ a basis of $\mathbb{R}^3$?

Our set has 3 vectors and

$$\dim \mathbb{R}^3 = 3$$

General Definition of dimension

$$\dim \mathbb{R}^n = n$$

Next, we check linear independence

$$\begin{bmatrix}1 & 0 & 1\\2 & 1 & 0\\0 & 1 & 3\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 1 & 3\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 5\end{bmatrix}$$

*Recall: All pivots for **nxn** matrix means **linear independence**

Theorem 33. *A basis is a minimal spanning set of V; that is the elements of the basis span V but you cannot delete any of these elements and still get all of V.*

Basis and Dim of four subspaces:

Rank [r] : Number of pivots matrix has

Let A be an $m \times n$ matrix with rank r

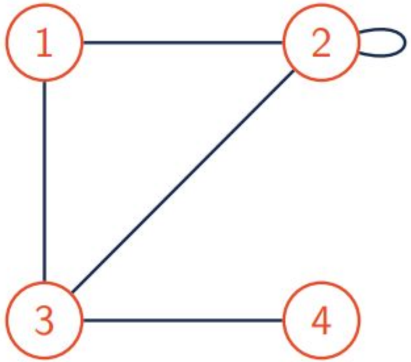
- dim Nul(A) = n - r
- dim Col(A) = r
- dim Nul(A^T) = m - r
- dim Col(A^T) = r

Graphs and Adjacency Matrices

A **graph** is a set of nodes (or: vertices) that are connected through edges.

Definition. Let $\mathcal{G}$ be a graph with n nodes. The **adjacency matrix** of $\mathcal{G}$ is the $n \times n$-matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between node } i \text{ and node } j \\ 0 & \text{otherwise .} \end{cases}$$



N1 N2 N3 N4

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Node 1: Connected to N2 & N3

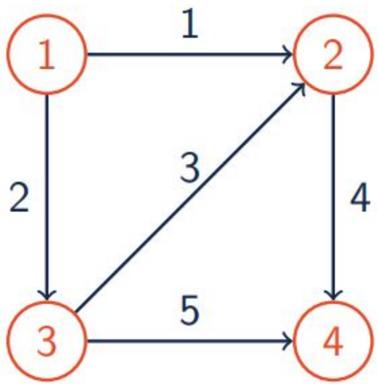
Node 2: Connected to N1, N2, & N3

Node 3: Connected to N1, N2 & N4

Node 4: Connected to N3

Directed Graphs

Definition. A **directed graph** is a set of vertices connected by edges, where the edges have a direction associated with them.



N1 N2 N3 N4

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Node 1: Nothing pointing to N1

Node 2: N1 and N3 pointing to N2

Node 3: N1 points to N3

Node 4: N2 and N3 pointing to N4

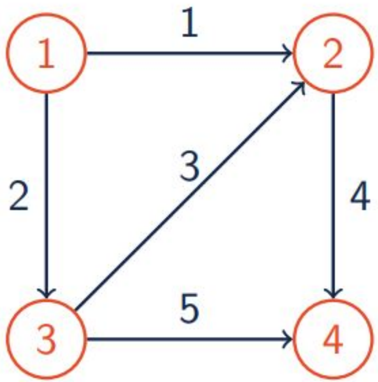
Definition. Let G be a directed graph with m edges and n nodes. The **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} 1, & \text{if there is a directed edge from node } j \text{ to node } i \\ 0, & \text{otherwise} \end{cases}$$

Edge-Node Incidence

Definition. Let G be a directed graph with m edges and n nodes. The **edge-node incidence matrix** of G is the $m \times n$ matrix $A = (a_{i,j})_{i,j}$ with

$$a_{i,j} = \begin{cases} -1, & \text{if edge } i \text{ leaves node } j \\ +1, & \text{if edge } i \text{ enters node } j \\ 0, & \text{otherwise} \end{cases}$$



N1	N2	N3	N4	
−1	1	0	0	Edge 1: Leaves N1; Enters N2
−1	0	1	0	Edge 2: Leaves N1; Enters N3
0	1	−1	0	Edge 3: Leaves N3; Enters N2
0	−1	0	1	Edge 4: Leaves N2; Enters N4
0	0	−1	1	Edge 5: Leaves N3; Enters N4

Orthogonal Complements

Definition. Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is the subspace $W^\perp$ of all vectors that are orthogonal to W; that is

$$W^\perp := \{\mathbf{v} \in \mathbb{R}^n \ : \ \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

Some helpful theorems:

- $(W^\perp)^\perp = W$
- $\text{Nul}(A) = \text{Col}(A^T)^\perp$
- $\text{Nul}(A)^\perp = \text{Col}(A^T)$
- $\text{Nul}(A^T) = \text{Col}(A)^\perp$

Theorem 43. *Let V be a subspace of $\mathbb{R}^n$. Then $\dim V + \dim V^\perp = n$.*

Coordinates

Standard basis (ε):

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Generally, if $\mathbf{v}_1$, $\mathbf{v}_2$, … $\mathbf{v}_p$ are a basis B of vector space V, the coordinate vector of any vector **w** in V is:

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \qquad \text{if } \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

This coordinate vector is unique!

Coordinate Example

Let $V = \mathbb{R}^2$, and consider the bases

$$\mathcal{B} := \left(\mathbf{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$\mathcal{E} := \left(\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}\right)$$

$\mathbf{w} = \begin{bmatrix}3\\-1\end{bmatrix}$. Determine $\mathbf{w}_\mathcal{B}$ and $\mathbf{w}_\mathcal{E}$

We want to find '**w**' in terms of **B**'s and **E**'s coordinate planes

$$\begin{bmatrix}3\\-1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = 1\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\mathbf{w}_\mathcal{B} = \begin{bmatrix}1\\2\end{bmatrix}$$

This is what [3,-1] looks like in 'basis' **B**

$$\begin{bmatrix}3\\-1\end{bmatrix} = c_1\begin{bmatrix}1\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathbf{w}_\mathcal{E} = \begin{bmatrix}3\\-1\end{bmatrix}$$

This is what [3,-1] looks like in 'basis' **E**

Orthogonal and Orthonormal Bases

Definition. An **orthogonal basis** (an **orthonormal basis**) is an orthogonal set of vectors (an orthonormal set of vectors) that forms a basis.

Theorem 47. *Let $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be an orthogonal basis of $\mathbb{R}^n$, and let $\mathbf{v} \in \mathbb{R}^n$. Then*

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \ldots + \frac{\mathbf{v} \cdot \mathbf{b}_n}{\mathbf{b}_n \cdot \mathbf{b}_n}\mathbf{b}_n.$$

When $\mathcal{B}$ is orthonormal, then $\mathbf{b}_i \cdot \mathbf{b}_i = 1$ for $i = 1, \ldots, n$.

Theorem 48. *Let $\mathcal{U} = (\mathbf{u}_1, \ldots, \mathbf{u_n})$ be an orthonormal basis of $\mathbb{R}^n$. Then*

$$I_{\mathcal{U},\mathcal{E}_n} = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{bmatrix}^T.$$

Why? An $n \times n$-matrix Q is **orthogonal** if $Q^{-1} = Q^T$

In-Person Resources

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Math 257	12pm-6pm	12pm-2pm 4pm-7pm 8pm-10pm	1pm-3pm 5pm-10pm	12pm-7pm	2pm-9pm	12pm-2pm 3pm-5pm	

Online Resources

Vector Spaces:
https://www.youtube.com/watch?v=XDvSsDsLVLs&ab_channel=TrevTutor

Linear Combinations, Spans, and Basis Vectors:
https://www.youtube.com/watch?v=k7RM-ot2NWY&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=2&ab_channel=3Blue1Brown

Change of Basis:
https://www.youtube.com/watch?v=P2LTAUO1TdA&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=13&ab_channel=3Blue1Brown

Abstract Vector Spaces:
https://www.youtube.com/watch?v=TgKwz5Ikpc8&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&index=16&ab_channel=3Blue1Brown

Questions?

Join the queue to see the worksheet and slides!

Important Definitions

Rank - Nullity Theorem for m x n matrix: $dim(col(A)) + dim(null(A)) = n$

Linearity: $T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$

Dimension:

$$dim(col(A)) = rank = \#pivot\ vars.$$
$$dim(null(A)) = nullity = \#free\ vars.$$

Subspace Criterion: Closed under linear combination, defined as the span of basis vectors

Linear Independence:

$$x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n} = \vec{0}$$
$$\text{iff } x_1 = x_2 = \cdots = x_n = 0$$

Orthogonal Complements:

$$col(A)^\perp = null(A^\top)$$
$$col(A^\top)^\perp = null(A)$$

Coordinate Inverse: $I_{\mathcal{A}\mathcal{E}} = I_{\mathcal{E}\mathcal{A}}^{-1}$