



The Grainger College of Engineering

Center for Academic Resources in Engineering

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# MATH 241

Midterm 2 Review

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Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

## QR Code to the Queue



The queue contains the worksheet and the solution to this review session

# Vector Functions

- $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$
- the x, y, z positions are each described by a function

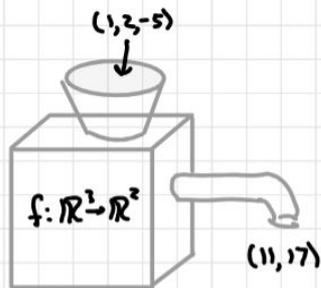
Derivative of vector function  $\rightarrow$  take derivative of each component w/ respect to t

Velocity  $\rightarrow$  speed and direction

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \langle f'(t), g'(t), h'(t) \rangle$$

# Functions of several variables

In general, consider  
 $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$



Examples:

1) Temperature at a point  
 $(x, y, z)$  in this room

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}$$

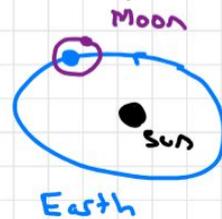
(Chapter 14)

2) Position of moon  
at time  $t$  (relative to sun)

$$r: \mathbb{R} \rightarrow \mathbb{R}^3 \quad (\text{parametric curve})$$

$$r(t) = (x(t), y(t), z(t))$$

(Chapter 13)

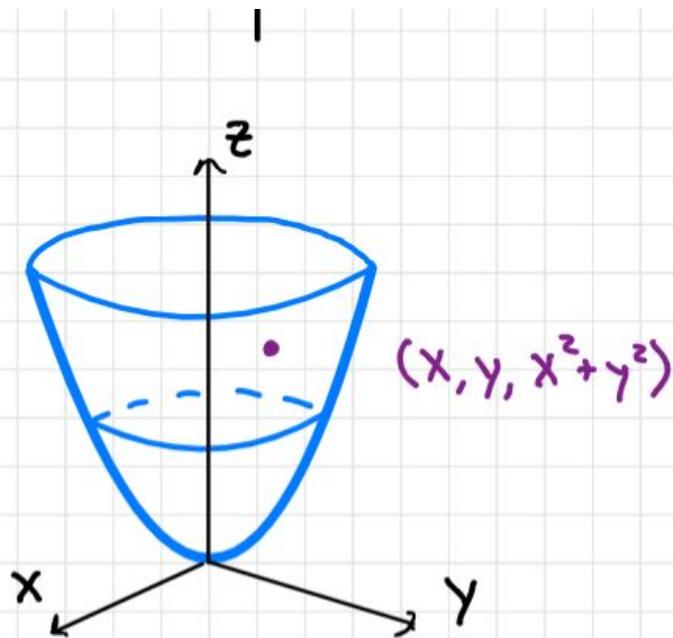


# Functions of several variables

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\text{e.g. } f(x,y) = x^2 + y^2$$

Graph of  $f$  is set of  
points  $(x, y, f(x, y)) \in \mathbb{R}^3$



$$\cos(\theta) \cos(\varphi) - \sin(\theta) \sin(\varphi) = \cos(\theta + \varphi)$$

# Space Curves

- Curve is 1-dimensional object
- Parametrization is how you draw it

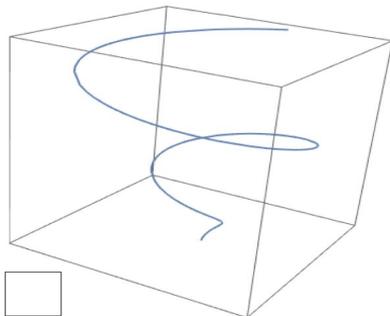
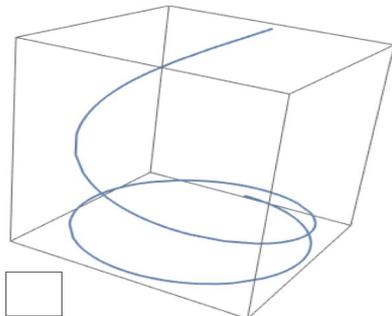
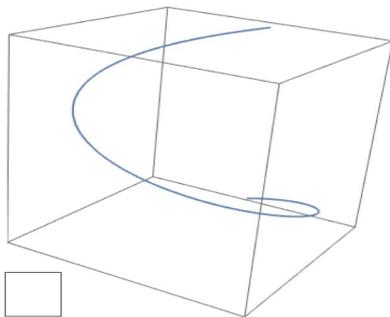
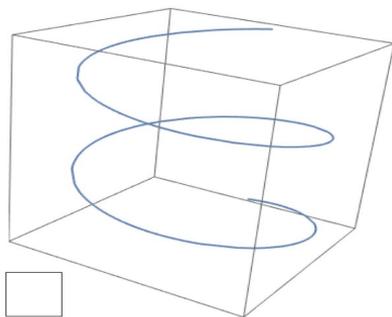
$r(t)$  = position  $r'(t)$  = velocity  $r''(t)$  = acceleration  $|r'(t)|$  = speed

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



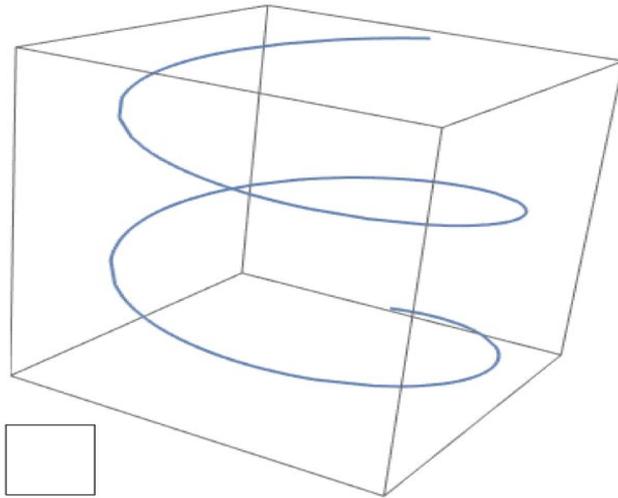
## Example Question #2

Let  $C$  be the curve parameterized by  $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$  for  $0 \leq t \leq 2\sqrt{\pi}$ . Check the corresponding picture of  $C$ .



## Example Solution #2

$$\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle \text{ for } 0 \leq t \leq 2\sqrt{\pi}$$



## Example Question #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface  $z = xy$ .

## Example Solution #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface  $z = xy$ .

$$\overrightarrow{r}_{\text{cyl}} = \langle 4\cos t, 4\sin t \rangle$$

$$z = xy = 16\cos t \cdot \sin t$$

$$\vec{r}(t) = \langle 4\cos t, 4\sin t, 16\cos t \cdot \sin t \rangle$$

# Limits

- When computing multivariable limits,
  - Check **multiple paths** (lines and power functions) to see if there are conflicting values. If so, limits DNE
  - **Factor** (difference of squares)
  - Use **polar coordinates**
  - Try **squeeze theorem**

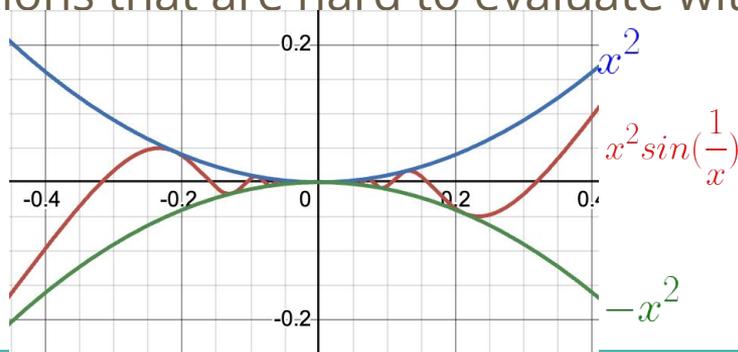
$$r = \sqrt{x^2 + y^2}$$

$$x = r \cdot \cos\theta$$

$$y = r \cdot \sin\theta$$

# Squeeze Theorem

- We have three functions such that near  $x$ :  $f(x) \leq g(x) \leq h(x)$
- If  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$ , then  $\lim_{x \rightarrow a} g(x) = L$
- Great to use for functions that are hard to evaluate with limit laws



# Limit Laws

## Sum Law

## Difference Law

## Constant Multiple Law

## Product Law

## Quotient Law

1. The limit of a sum is the sum of the limits.
2. The limit of a difference is the difference of the limits.
3. The limit of a constant times a function is the constant times the limit of the function.
4. The limit of a product is the product of the limits.
5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

# Continuity

- A function  $f(x,y)$  is continuous at point  $(x,y)$  if

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

- If this holds for all points  $(a,b)$ , then the function is continuous over the 2D plane.

# Partial Derivatives

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

$$f(x, y) \quad \Rightarrow \quad f_x(x, y) = \frac{\partial f}{\partial x} \quad \& \quad f_y(x, y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (f_y) = (f_y)_y = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (f_y) = (f_y)_x = f_{yx}$$

## Arc Length Formula

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

# Linear Approximation & Tangent planes

- If  $z = f(x, y)$  and  $f$  is **differentiable** at  $(a, b)$ , then the value of  $f(m, n)$  can be approximated by

$$f(m, n) \approx L(m, n)$$

$$L(m, n) = f(a, b) + f_x(a, b) \cdot (m - a) + f_y(a, b) \cdot (n - b)$$

## Example Question #5

- Compute the following limits

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^2 + y^8}$$

$$\lim_{x \rightarrow 0} x^3 \cos\left(\frac{2}{x}\right)$$

$$\lim_{(x,y) \rightarrow (-1,0)} \frac{x^2 + xy + 3}{x^2y - 5xy + y^2 + 1}$$

- Determine whether the following function is continuous at  $(0, 0)$

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

## Example Solution #5

- Compute the following limits

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = 0 \quad (\text{Use polar coordinates})$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^2 + y^8} = \text{DNE} \quad (\text{Check } x = y^4 \text{ and } x = -y^4)$$

$$\lim_{x \rightarrow 0} x^3 \cos\left(\frac{2}{x}\right) = 0 \quad (\text{Squeeze Theorem})$$

$$\lim_{(x,y) \rightarrow (-1,0)} \frac{x^2 + xy + 3}{x^2y - 5xy + y^2 + 1} = 4 \quad (\text{Plug in } (-1, 0) \text{ directly})$$

## Example Solution #5

- Determine whether the following function is continuous at  $(0, 0)$

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

**On line  $y = x$ ,  $f(x, y) = 1/3$  at any point except  $(0, 0)$ . Since there is a discontinuity at  $(0, 0)$ , the function is not continuous.**

# Velocity and Acceleration

$$\vec{r}'(t) = \vec{v}(t) \rightarrow \vec{r}(t) = \int \vec{v}(t) dt$$

$$\vec{v}'(t) = \vec{a}(t) \rightarrow \vec{v}(t) = \int \vec{a}(t) dt$$