Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 241 – Calculus III

Midterm 2 CBTF Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Mar. 1, 5-7 pm Meredith and Lydia

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/845
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Consider the vector function $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$. Find the length from $(2, 0, 0)$ to $(2, 0, 4\pi)$ of the vector function.

Need to figure out the values of t at start and end point

a: $(2, 0, 0) \rightarrow$ z = 0 = t $\rightarrow$ t = 0

b: $(2, 0, 4\pi) \rightarrow$ z = 4π = t $\rightarrow$ t = 4π

$L = \int_0^{4\pi} \sqrt{(-2sint)^2 + (2cost)^2 + 1^2}dt = \int_0^{4\pi} \sqrt{4sin^2t + 4cos^2t + 1}dt = \int_0^{4\pi} \sqrt{5}dt = 4\sqrt{5}\pi$

2. Calculate the following derivatives:

(a) Find $\frac{\mathrm{d}f}{\mathrm{d}t}$ for $f(x, y) = xe^{xy}$, $x(t) = t^2$, $y(t) = \frac{1}{t}$

(b) Find f_t for $f(x, y) = 2xy$, $x(s, t) = st$, $y(s, t) = s^2t^2$

(a) Applying the chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\partial f}{\partial x} = e^{xy} + x(ye^{xy}) \qquad \frac{dx}{dt} = 2t$$

$$\frac{\partial f}{\partial y} = x^2e^{xy} \qquad \frac{dy}{dt} = -\frac{1}{t^2}$$

$$\frac{df}{dt} = \left[e^{xy} + x(ye^{xy})\right](2t) + [x^2e^{xy}]\left(-\frac{1}{t^2}\right)$$

$$\boxed{\frac{df}{dt} = 2te^t + t^2e^t}$$

(b) Applying Chain Rule:

$$f_t = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial f}{\partial x} = 2y,\ \frac{\partial x}{\partial t} = s,\ \frac{\partial f}{\partial y} = 2x,\ \frac{\partial y}{\partial t} = 2ts^2$$

$$\boxed{f_t = 6s^3t^2}$$

3. What is the partial derivative of $f(x, y, z) = e^x \sin(yz) z^3 \ln(y)$ with respect to x.
It's the same function.

4. Consider a function $f(t) = f(x(t), y(t), z(t)) = xyz - z^2$, where $x(t)$, $y(t)$, $z(t)$ are defined as followed:

$$x(t) = 2t^2 + 1 \qquad y(t) = 3 - \frac{1}{t} \qquad z(t) = 3$$

Find the following values:

(a) $f_z(3, 1, 2)$ (b) $\frac{dx}{dt}\Big|_{(t=0)}$ (c) $\frac{df}{dt}\Big|_{(t=1)}$

(a)

$$\frac{\partial}{\partial z}(xyz - z^2)$$

$$xy - 2z$$

$$\boxed{f_z(3, 1, 2) = 3 - 4 = -1}$$

(b)

$$\frac{d}{dt}x = 4t$$

$$\boxed{\frac{dx}{dt}\Big|_{(t=0)} = 0}$$

(c)

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = yz(4t) + xz\left(\frac{1}{t^2}\right) + (xy - 2z)(0)$$

When $t = 1$:
$x = 3$
$y = 2$
$z = 3$

Plug into the equation:

$$\boxed{\frac{df}{dt}\Big|_{(t=1)} = 24 + 9 + 0 = 33}$$

5. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4}$$

Does this limit exist? If so, what is its value? Justify your answer.

Along $x = 0$

$$\lim_{(x,y)\to(0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4} = \lim_{(x,y)\to(0,0)} \frac{y^4}{y^4} = 1$$

Along $x = y$

$$\lim_{(x,y)\to(0,0)} \frac{y^4 \cos^2 x}{x^4 + y^4} = \lim_{(x,y)\to(0,0)} \frac{\cos^2 x}{2} = \frac{1}{2}$$

Since $0 \neq 1/2$, the $\boxed{\text{Limit DNE}}$

6. If $f(x, y, z) = xye^z$, find the gradient of f and the directional derivative at (2,5,0) in the direction of $\vec{v} = 2\hat{i} - \hat{j} + \hat{k}$.

$f_x = ye^z$ $\qquad$ $f_y = xe^z$ $\qquad$ $f_z = xye^z$

The gradient of f is $\boxed{\nabla f(x, y, z) = \langle ye^z, xe^z, xye^z\rangle}$

The unit vector of $\vec{v}$ is

$$\vec{u} = \langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$$

At (2,5,0),

$$\nabla f(2, 5, 0) = \langle 5, 2, 10\rangle$$

The directional derivative at (2,5,0) in the direction of $\vec{v}$ is calculated to be

$$D_u f(2, 5, 0) = \nabla f(2, 5, 0) \cdot \vec{u} = \langle 5, 2, 10\rangle \cdot \langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \rangle$$

$$= \frac{18}{\sqrt{6}} = \boxed{3\sqrt{6}}$$

7. Evaluate the following functions with $\lim_{(x,y)\to(0,0)}$:

$$(a) f(x,y) = \frac{3xy - x^2y}{x^2+y^2+xy} \qquad (b) f(x,y) = \frac{y\sin(x) + y^2e^x}{y} \qquad (c) f(x,y) = \frac{\left(x^2+y^2\right)^5}{x^{10}+y^4}$$

(a) Use polar coordinates:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta)$$

$$\lim_{r\to 0} \frac{3r^2\sin\theta\cos\theta - r^3\cos^2\theta\sin\theta}{r^2(\cos^2\theta + \sin^2\theta + \cos\theta\sin\theta)}$$

$$\lim_{r\to 0} \frac{3\sin\theta\cos\theta - r\cos^2\theta\sin\theta}{(1+\cos\theta\sin\theta)}$$

Let $r \to 0$ (eliminating the $\cos(\theta)^2\sin(\theta)$ term). We end up with

$$\frac{3\sin\theta\cos\theta}{(1+\cos\theta\sin\theta)}$$

Plug in $\theta = 0$ and $\theta = \frac{\pi}{4}$

$$\theta = 0 \qquad \frac{0}{1} = 0$$

$$\theta = \frac{\pi}{4} \qquad \frac{\frac{3}{2}}{1+\frac{1}{2}} = 1$$

The limit has different values for different θ, thus the limit $\boxed{\text{DOES NOT EXIST}}$

(b) Divide out a y, and the let $(x,y) \to (0,0)$

$$\lim_{(x,y)\to(0,0)} \sin(x) + ye^x \to \boxed{0}$$

(c) Take the path $x = 0$ and you end up with $f(0,y) = y^6$, which has a limit of 0
Take the path $y = 0$ and you end up with $f(x,0) = \frac{x^{10}}{x^{10}}$ which has a limit of 1

Thus, the limit $\boxed{\text{DOES NOT EXIST}}$

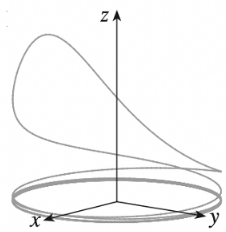
Match the vector functions below to the corresponding graph.

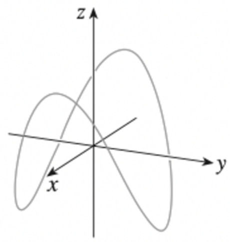
8. (i) $x = t, y = 1/(1+t^2), z = t^2$

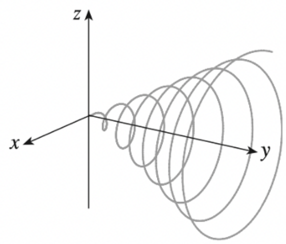
(ii) $x = \cos t, y = \sin t, z = \cos 2t$

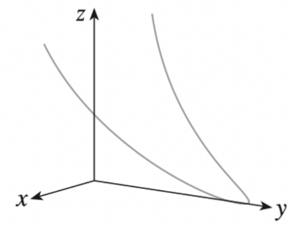
(iii) $x = \cos t, y = \sin t, z = 1/(1+t^2)$

(iv) $x = t\cos t, y = t, z = t\sin t, t \geq 0$









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(III), (II), (IV), (I)

Tricks: If (x, y, z) are all sin/cos → close trace

Coefficients in sin/cos → more curves along that axis (see II)

If only 2 sin/cos → circular path around the sin/cos axis (see III)

If multiplied by t, radius increases.

Take x = 0 / y = 0 / z = 0 to examine cross sectional curves.

9. Find the limits for the following expressions:

(i)
$$\lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$$

(ii)
$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$

(iii)
$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$

(iv)
$$\lim_{(x,y)\to(0,0)} \frac{x^6+x^2y^4}{x^2+y^2}$$

(i) Plug the point into the function.

$$= \frac{(-1)^2+(-1)(0)+3}{(-1)^2\cdot(0)-5(-1)(0)+(0)^2+1} = 4$$

(ii) Polar coordinates! As r approaches 0, the whole function approaches 0

$$\frac{r^3\cos^3(\theta)}{r^2} = r\cos^3(\theta) \to 0$$

(iii) Test two different lines.
Test path

$$x = y^4$$
$$= \frac{y^8}{y^8+y^8} = \frac{1}{2}$$

Test path x = 0

$$= \frac{(0)\,y^4}{0^2+y^8} = 0$$

These limits are not equal, so DNE!

(iv) More polar coordinates!

$$= \frac{r^6\cos^6(\theta)+r^2\cos^2(\theta)\cdot r^4\sin^4(\theta)}{r^2}$$
$$= r^4\cos^6(\theta)+\cos^2(\theta)\cdot r^4\sin^4(\theta)$$
$$= r^4\big(\cos^6(\theta)+\cos^2(\theta)\sin^4(\theta)\big) = 0$$

As r approaches 0, the whole function approaches 0.

10. Find the derivatives of the function $f(x, y) = x^2 + y^2 + xy + y^3$.

(i) f_x, f_y

(ii) Total differential

(iii) f_{xx}, f_{yy}

(iv) f_{xy}, f_{yx}

(i) $f_x = 2x + 0 + y + 0 = 2x + y$

$f_y = 0 + 2y + x + 3y^2 = 2y + x + 3y^2$

(ii) $df = \frac{df}{dx}dx + \frac{df}{dy}dy = (2x + y)dx + (2y + x + 3y^2)dy$

(iii) $f_{xx} = \frac{d}{dx}(f_x) = 2 + 0 = 2$

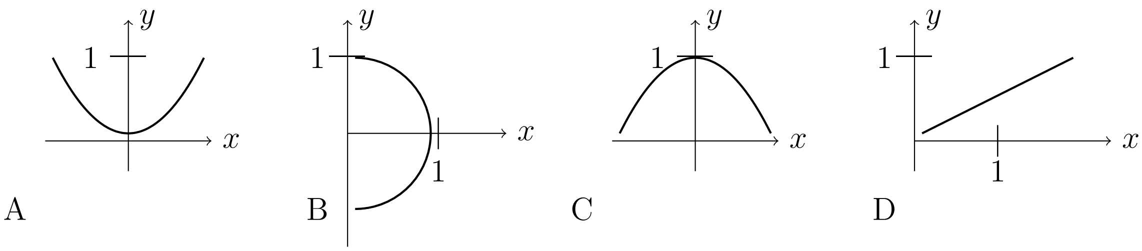
$f_{yy} = \frac{d}{dy}(f_y) = 2 + 0 + 6y = 2 + 6y$

(iv) $f_{xy} = \frac{d}{dy}(f_x) = 0 + 1 = 1$

$f_{yx} = \frac{d}{dx}(f_y) = 0 + 1 + 0 = 1$

$\rightarrow f_{xy} = f_{yx}$, function is continuous.

11. Let $r(t) = \langle \sin(t), \cos^2 t \rangle, 0 \leq t \leq 2\pi$. Which graph below represents this curve?



Converting this into a Cartesian equation gives

$$y + x^2 = 1 \rightarrow y = 1 - x^2$$

Which is a concave down parabola with its vertex at $(0, 1)$ shown in graph C.

12. Assume you are walking around the surface of a spherical planet with a radius of 2. If you are walking clockwise on the xy-plane, what is the parametrization of the path after circling it twice?

The equation for a sphere (with radius 2) is $x^2 + y^2 + z^2 = 4$. On the xy-plane, $z = 0$.
The equation then becomes $x^2 + y^2 = 4$, which yields the parametrization $\boxed{\langle 2\cos t, -2\sin t, 0\rangle}$ for the clockwise direction.
2 rounds around the planet means that the domain of t is $\boxed{0 \leq t \leq 4\pi}$.
Since there is no restraint on where the curve starts, the answer can also be $\langle 2\sin t, 2\cos t, 0\rangle$ or $\langle -2\cos t, 2\sin t, 0\rangle$.