MATH 285
Midterm 1 Review

CARE

Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.

Topics

I. Classifying Differential Equations
II. Slope Fields
III. Existence and Uniqueness
IV. Autonomous Equations
V. Solving Methods:
   I. Separable
   II. Exact
   III. Integrating Factor

Differential Equations

- "A differential equation is any relationship between a function (usually denoted y(t)) and its derivatives up to some order."
- **Slope Fields:** Help visually model a differential equation
  - Lines parallel to the derivative at each point
  - Can show overall direction and shape of the solution, as well as equilibrium values

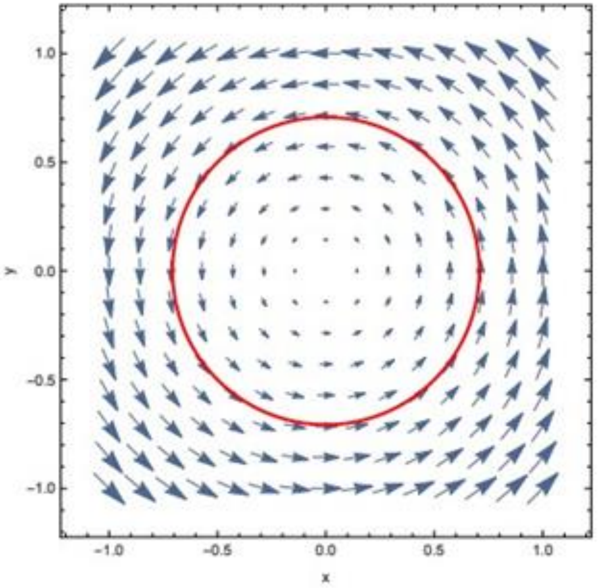

Figure 1.4: A slope field for $\frac{dy}{dt} = -\frac{t}{y}$ (blue) together with a solution curve (red).

Differential Equations. Bronski J., Manfroi A., Figure 1.4

Classifications

Linear

2nd order

Ordinary

$$\frac{d^2y}{dt^2} + \sin(t)\frac{dy}{dt} + 15y = e^t$$

Ordinary vs Partial

- ODE's involve only standard derivatives
- PDE's involve partial derivatives

Linear vs Nonlinear

- Linear differential equations only have linear terms of the function and its derivatives
- Nonlinear equations are everything else

Order

- The order of a differential equation is the degree of the highest derivative it contains

Existence and Uniqueness Theorem

$$\frac{dy}{dt} = f(y, t) \qquad\qquad y(t_0) = y_0$$

- A solution to the differential equation is **guaranteed to exist** in the **interval** in which the **first derivative is continuous around the initial value**
- That solution is **guaranteed to be** unique if $\frac{\partial f(y,t)}{\partial y}$ is also **continuous around the initial value**

Autonomous Equations

- Autonomous equation: **does not explicitly involve independent variable**

$$\frac{dy}{dt} = f(y)$$

- Draw a **phase line**, identify points where the **derivative is 0**, and then **identify equilibria**
- Types of equilibria:
  - **Stable**: nearby points converge to the equilibrium
  - **Semi-stable**: points converge from one direction
  - **Unstable**: points diverge away from the equilibrium

Separable Equations

- Separable Equations are in the form:

$$\frac{dy}{dt} = f(y)g(t)$$

- Separable equations can be solved **directly through integration**

$$\int \frac{dy}{f(y)} = \int g(t)dt + C$$

Separable Equations Example

- Solve the following differential equation:

$$\frac{dy}{dt} = e^{-y/t} + \frac{y}{t}$$

- Notice this equation contains $\frac{y}{t}$, which suggests a homogenous substitution of $v = \frac{y}{t}$

Separable Equations Example

- From $v = \frac{y}{t}$, we can say:

$$y = vt$$

- Then we differentiate to get:

$$\frac{dy}{dt} = t\frac{dv}{dt} + v$$

Separable Equations Example

- Then we rewrite the original equation as:

$$t\frac{dv}{dt} + v = e^{-v} + v$$

- Simplifying gives us:

$$t\frac{dv}{dt} = e^{-v}$$

- Which can be separated as:

$$e^{v}\,dv = \frac{1}{t}\,dt$$

Separable Equations Example

- Integrating:

$$\int e^v \, dv = \int \frac{1}{t} \, dt$$

$$e^v = \ln|t| + C$$

- Solving for v:

$$v = \ln(\ln|t| + C)$$

- And plugging in $v = \frac{y}{t}$:

$$\frac{y}{t} = \ln(\ln|t| + C)$$

Separable Equations Example

- Final Answer:

$$\boxed{y = t\ \ln(\ln|t| + C)}$$

Exact Equations

- Exact equations have the form of:

$$\boldsymbol{N(x,y)\cdot y' + M(x,y) = 0}$$

- An equation is exact if the **partial derivatives of the two coefficient terms are equal**:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Solving Exact Equations

1. **Partially integrate** either *N* or *M*: $\int N \partial y$ **or** $\int M \partial x$

2. **Set equal to** Ψ + a constant of integration function: $\Psi = \int N \partial y + f(x)$ **or** $\Psi = \int M \partial x + f(y)$

3. Take the **derivative** with respect to the **opposite variable:** $\frac{\partial \Psi}{\partial x}$ **or** $\frac{\partial \Psi}{\partial x}$

4. **Set equal** to the **other term** you didn't integrate: $\frac{\partial \Psi}{\partial x} = M$ **or** $\frac{\partial \Psi}{\partial x} = N$

5. **Integrate** to solve for f(x) or f(y) and plug back into step 2 $\int f'(x)dx$ **or** $\int f'(y)dy$

Exact Equation Example

Solve the following differential equation:

$$(5x^2y + 2x + 4)\frac{dy}{dt} + (5xy^2 + 2y + 7) = 0$$

Exact Equation Example

The two terms are:

$$M(x,y) = 5xy^2 + 2y + 7$$

$$N(x,y) = 5x^2y + 2x + 4$$

In this example, we choose M to integrate (w.r.t x)

$$\psi(x,y) = \int (5xy^2 + 2y + 7)\, dx$$

$$\psi(x,y) = \frac{5}{2}x^2y^2 + 2xy + 7x + h(y)$$

Exact Equation Example

Then differentiate with respect to y:

$$\psi_y = 5x^2y + 2x + h'(y)$$

And set this equal to N:

$$5x^2y + 2x + h'(y) = 5x^2y + 2x + 4$$

$$h'(y) = 4$$

$$h(y) = 4y$$

Exact Equation Example

Final Answer:

$$\boxed{\frac{5}{2}x^2y^2 + 2xy + 7x + 4y = C}$$

Integrating Factor Method

1. Make sure your equation looks like: $$\frac{dy}{dt} + p(t)y = q(t)$$

2. Calculate the **integrating factor:** $$\mu(t) = e^{\int p(t)dt}$$

3. **Multiply the entire equation** by the integrating factor: $$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)q(t)$$

4. Re-write the left-hand side as the **result of product rule:** $$\frac{d}{dt}(\mu(t)y) = \mu(t)q(t)$$

5. **Integrate both sides** and rearrange to solve for *y(t)* $$\mu(t)y = \int \mu(t)q(t)dt$$

Integrating Factor Example

Solve the following differential equation:

$$y' + 3y = 2$$

Integrating Factor Example

The integrating factor is:

$$\mu(x) = e^{\int P(x)\,dx}$$

$$\mu(x) = e^{\int 3\,dx} = e^{3x}$$

So multiply both sides by the integrating factor:

$$e^{3x}y' + 3e^{3x}y = 2e^{3x}$$

The left side then becomes:

$$\frac{d}{dx}(e^{3x}y)$$

Integrating Factor Example

Then integrate both sides:

$$\frac{d}{dx}(e^{3x}y) = 2e^{3x}$$

$$e^{3x}y = \int 2e^{3x}\,dx$$

$$e^{3x}y = \frac{2}{3}e^{3x} + C$$

$$y = \frac{2}{3} + Ce^{-3x}$$

And solve for y:

$$y = \frac{2}{3} + Ce^{-3x}$$

Integrating Factor Example

Final Answer:

$$\boxed{y = \frac{2}{3} + Ce^{-3x}}$$

Thanks for Coming!