Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 285 – Intro Differential Equations

Midterm 1 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Febuary 18 from 4:00-5:50 pm in 433 Grainger Library - Regina, Sean

Session 2: Febuary 19 from 6:00-7:50 pm in 433 Grainger Library - Clive, Eric, Kimaya

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/846
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Consider the following differential equation and initial condition:

$$(16 - t^2)y' + t^3 y = \cos\left(\frac{t}{2}\right) \qquad y(1) = -5$$

On what interval is the unique solution certain to exist?

A) $(0, 2\pi)$

B) $(-2\pi, 0)$

C) $(-4, 0)$

D) $(-4, 4)$

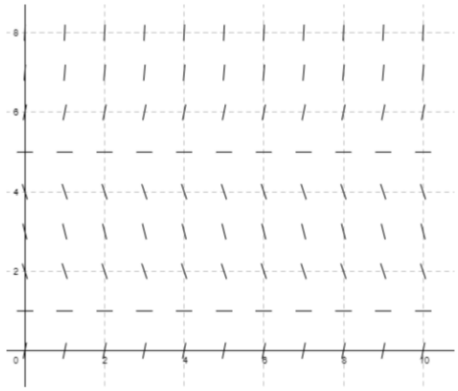
E) $(-\infty, -4)$

Put the differential equation in standard form:

$$y' + \frac{t^3}{(4-t)(4+t)}y = \cos\left(\frac{t}{2}\right)\frac{1}{(4-t)(4+t)}$$

The solution is certain to exist in one of the intervals of continuity of the coefficient functions. Those intervals are $(-\infty, -4)$, $(-4, 4)$ and $(4, \infty)$. Choose the interval containing the initial t-value, which is $t = 1$. So the answer is $\boxed{(-4, 4)}$

2. Which equation produces the direction field below?



A) $\frac{dy}{dx} = (x-1)(x-5)$

B) $\frac{dy}{dx} = xy$

C) $\frac{dy}{dx} = (y-1)(y-5)$

D) $\frac{dy}{dx} = (y+1)(y+5)$

E) $\frac{dy}{dx} = y^2$

Look for places where the slope is equal to zero. We can see from the graph that $\frac{dy}{dx} = 0$ at $y = 1$ and $y = 5$. The solution is **(C)**.

3. Which of the following equations are linear? Note: u_* denotes a first order partial derivative and u_{**} is a second order partial derivative.

(I) $\frac{d^2y}{dt^2} + e^y = 6t + 5$

(II) $(2t^3 + 6)\frac{d^5y}{dt^5} - \frac{d^3y}{dt^3} + 4y = t\cos(t - 1)$

(III) $u_y = uu_{xx} - u_{xy}$

(IV) $u_{xx} + xu_{xt} + t^2u_{tt} = \sin(x + 2t)$

A) (I), (II), (III)

B) (II) and (IV)

C) (I) and (IV)

D) (II)

E) (IV)

Linear DEs cannot have coefficients of the dependent variable or its derivatives. The answer is **(B)**.

4. What is the order of the following differential equation?

$$\cot(y)y''' + (t^2 + t + 9)y' - \ln\left(xy^2\right)y + 6y^9 = \sin\left(3t^5 + 1\right)$$

A) 9

B) 1

C) 3

D) 5

E) None

Highest derivative determines order. The answer is **(C)**.

5. Consider the following autonomous equation: $y' = (y^2 - 9)(5 - y)$
List the equilibrium solutions and classify them as stable, semi-stable, or unstable.

Equilibrium Points: $y' = 0$ at $y = -3, 3,$ and 5

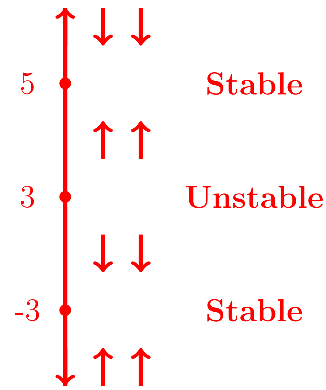
Draw a phase line to help determine stability.

Plug in $y > 5$, we get $y' < 0$

Plug in $3 > y > 5$, we get $y' > 0$

Plug in $-3 < y < 3$, we get $y' < 0$

Plug in $y < -3$, we get $y' > 0$



6. Prove that the following ODE is exact and find the general solution using the exact equations method with ψ: $(3x^2 - 3y^2)\frac{dy}{dx} + (3x^2 + 6xy) = 0$

First, identify $N(x,y)$ and $M(x,y)$: $N(x,y) = (3x^2 - 3y^2)$ and $M(x,y) = (3x^2 + 6xy)$
Next, take the respective partial dertivatives of $N_x(x,y)$ and $M_y(x,y)$.

$$\frac{\delta N}{\delta x} = 6x = \frac{\delta M}{\delta y} = 6x$$

The ODE has been proven to be exact, so we can apply the exact equations method. Knowing that $N(x,y) = \psi_y$ and $M(x,y) = \psi_x$, ψ can be found. Take the partial integral of $M(x,y)$ with respect to x:

$$\psi = \int (3\text{x}^2 + 6xy)dx = \quad x^3 + 3x^2y + C(y)$$

Now, we need to take the partial derivative of ψ with respect y, so we can set it equal to $N(x,y)$ to solve for $C(y)$.

$$\psi_y = 3x^2 + C'(y) = N(x,y)$$
$$3x^2 + C'(y) = 3x^2 - 3y^2$$
$$C'(y) = -3y^2$$
$$C(y) = \int -3y^2\, dy = -y^3 + C$$

Finally, by subbing our expression for $C(y)$ into our equation for ψ, our final solution is:

$$\psi(x,y) = x^3 + 3x^2y - y^3 + C$$

7. Solve $x(\frac{dy}{dx}) = 2y + x$

We rewrite the equation in the following form:

$$\frac{dy}{dx} - 2\frac{y}{x} = 1$$

Note that it is a 1st order linear equation
Computing the integrating factor:

$$\mu(x) = e^{-\int \frac{2}{x}dx} = e^{-2\ln(x)} = x^{-2}$$

Multiplying both sides by this integrating factor, we get [x^{-2}y]' = x^{-2}

Now we integrate both sides

$$\int [x^{-2}y]'dx = \int x^{-2}dx$$

We get $x^{-2}y = -x^{-1} + C$

$$\boxed{y(x) = -x + Cx^2}$$

8. Find the solution to the following initial value problem

$$(x^4+1)\left(\frac{dy}{dx}\right) = 2x^3y^2 \qquad y(0) = \left(\frac{3}{2}\right)$$

Using separation of variables, we can rewrite the equation in the following form:

$$\left(\frac{dy}{y^2}\right) = \left(\frac{2x^3}{x^4+1}\right)dx$$

Now integrating both sides with the use of u-sub on the right side:

$$\int y^{-2}dy = \frac{1}{2}\int \frac{1}{u}du$$

$$u = x^4+1$$
$$du = 4x^3dx$$
$$\frac{1}{2}du = 2x^3dx$$

Upon integration, the resulting equation can be written:

$$\frac{-1}{y} = \frac{1}{2}\ln\left(x^4+1\right) + c$$

Using algebraic manipulation, the equation can be further simplified:

$$-y = \frac{1}{\frac{1}{2}\ln(x^4+1)+c}$$

$$y = \frac{-1}{\frac{1}{2}\ln(x^4+1)+c}$$

Now by plugging in initial condition:

$$\frac{3}{2} = \frac{-1}{\ln(0^4+1)+c}$$

$$c = \frac{-2}{3}$$

Final solution with intial condition is:

$$\boxed{y = \frac{-1}{\frac{1}{2}\ln(x^4+1)-\frac{2}{3}}}$$