The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 1 Review

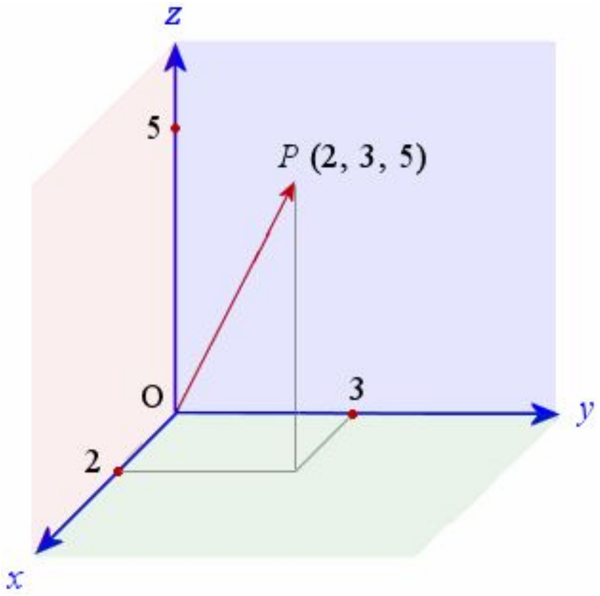
DISCLAIMER:
Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

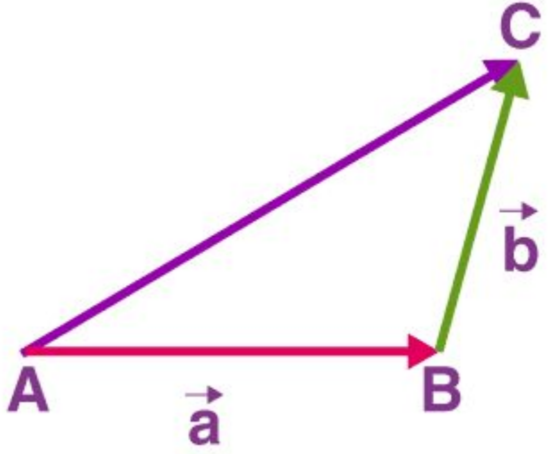
The queue contains the worksheet and the solution to this review session

Vectors

- Has both magnitude and direction



- Vectors are added “tip to tail”

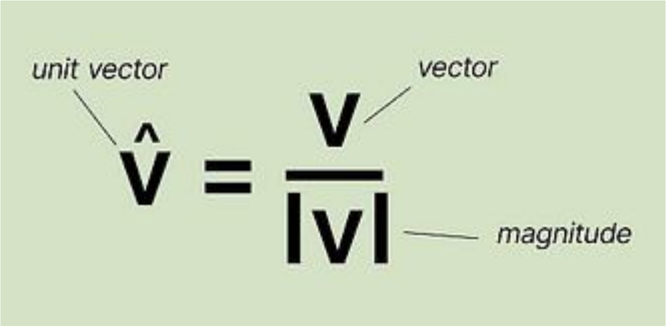


Magnitude of a Vector

- Magnitude/Length of Vector: |v|

$$\text{Magnitude} = \sqrt{x^2 + y^2 + z^2} \quad \text{(for 3D vectors)}$$

- Unit Vector



Dot Product

Dot product from components. #rvv-es

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Dot product from length/angle. #rvv-ed

$$\vec{a} \cdot \vec{b} = ab \cos\theta$$

Length and angle from dot product. #rvv-el

$$a = \sqrt{\vec{a} \cdot \vec{a}}$$

$$\cos\theta = \frac{\vec{b} \cdot \vec{a}}{ba}$$

Cross Product

Cross product in components. #rvv-ex

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\,\hat{\imath} + (a_3 b_1 - a_1 b_3)\,\hat{\jmath} + (a_1 b_2 - a_2 b_1)\,\hat{k}$$

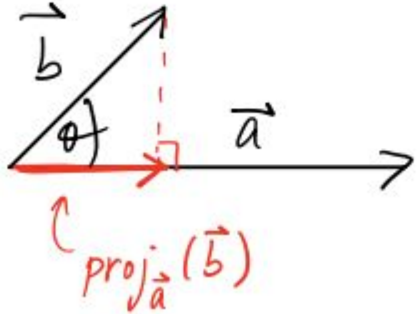
Cross products of basis vectors. #rvv-eo

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

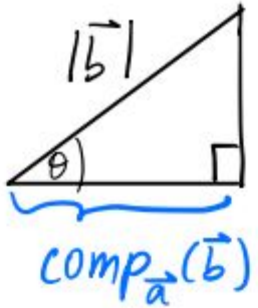
$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}$$

Projection and Components

$$\text{proj}_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|}$$



$$\text{comp}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

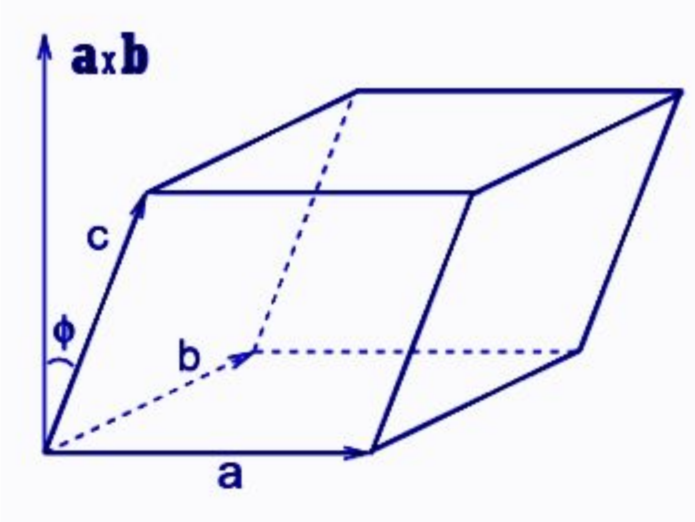


Scalar Triple Product

- A · (B x C)
- Represents the parallelepiped volume enclosed by the three vectors

$$\vec{A} = \langle a_1, a_2, a_3 \rangle, \ \vec{B} = \langle b_1, b_2, b_3 \rangle, \ \vec{C} = \langle c_1, c_2, c_3 \rangle$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$



Equations for Lines and Planes

- The equation for a line *L* on a plane can be parametrized:
  - Here, r_0 is a vector between the origin and a point on the plane
  - And *v* is a line on the plane

$$L = \vec{r_0} + t\vec{v}$$

$$\vec{r_0} = \; < x_0, \; y_0, \; z_0 >$$

$$\vec{v} = < v_1, v_2, v_3 >$$

$$\begin{Bmatrix} x(t) & = & x_0 + tv_1 \\ y(t) & = & y_0 + tv_2 \\ z(t) & = & z_0 + tv_3 \end{Bmatrix}$$

Equation of Plane

$$\vec{n} \cdot (Q - P) = 0$$

where $Q = (x, y, z)$,

$P = (x_0, y_0, z_0)$

$\vec{n}$ is the normal vector of the plane.

$$Ax + By + Cz = D$$

- Describes a plane in which *A*, *B*, and *C* are the components of the normal vector
- To find *D*, you need a point on the plane:

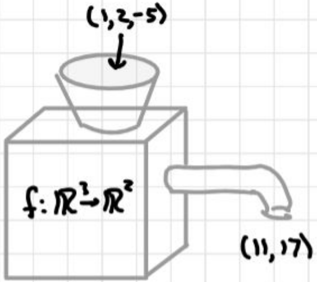
$$< x_0, y_0, z_0 >$$

$$D = Ax_0 + By_0 + Cz_0$$

Functions of several variables

In general, consider

$f: \mathbb{R}^n \to \mathbb{R}^m$



Examples:

1) Temperature at a point (x,y,z) in this room

$T: \mathbb{R}^3 \to \mathbb{R}$

(Chapter 14)

2) Position of moon at time t (relative to sun)

$r: \mathbb{R} \to \mathbb{R}^3$ (parametric curve)

$r(t) = (x(t), y(t), z(t))$

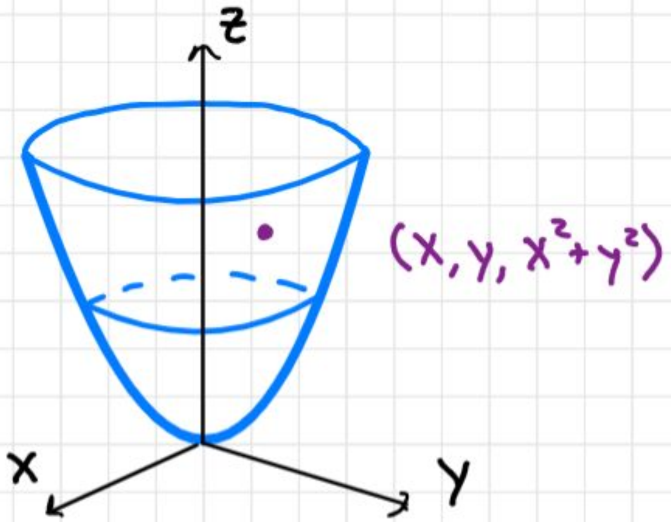
(Chapter 13)

Functions of several variables

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

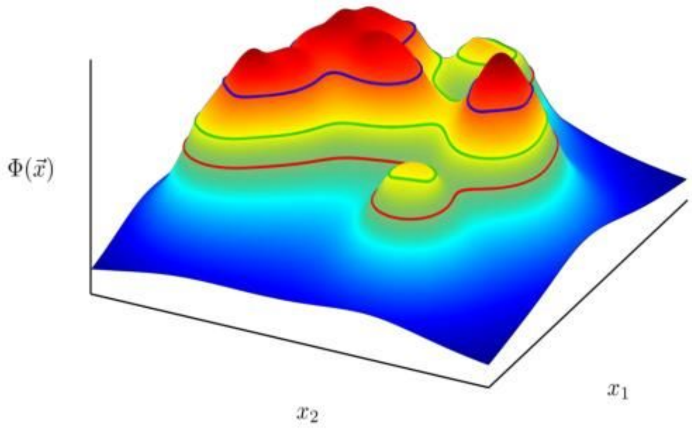
e.g. $f(x,y) = x^2 + y^2$

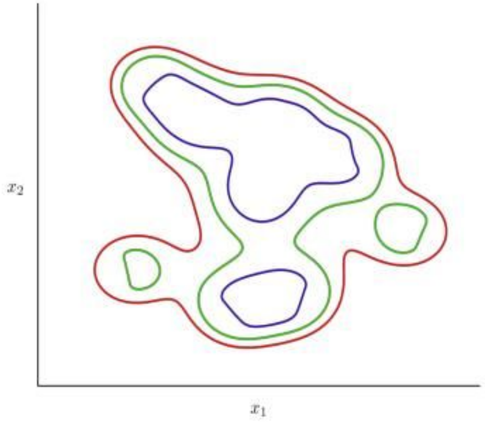
graph of f is set of points $(x, y, f(x,y)) \in \mathbb{R}^2$



Level Sets

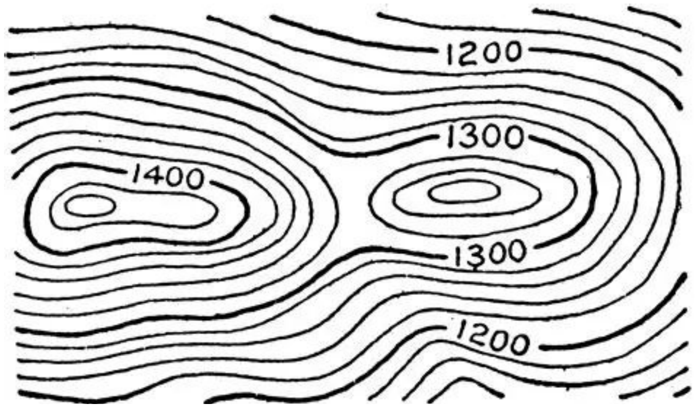
- Curve generated by “slicing” a multivariable function at a constant function value (height)

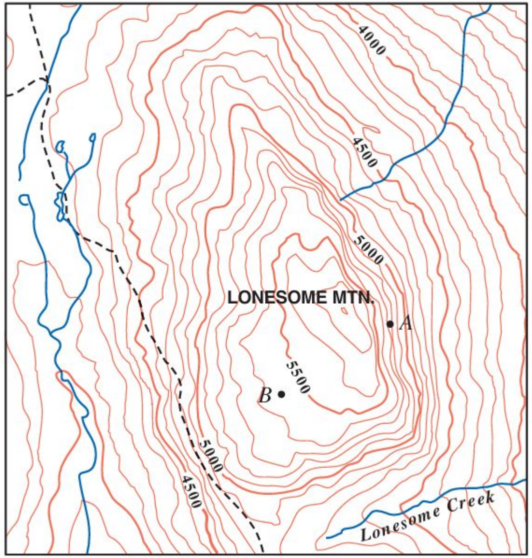




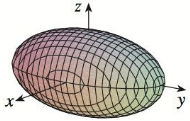
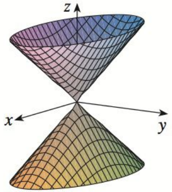
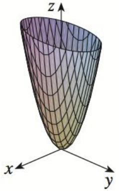
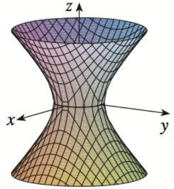
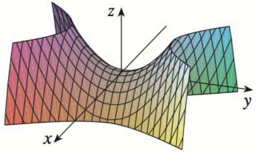
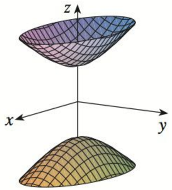
Contour Map

- Map of many level sets at different function values (heights)





Quadric Surface

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Example Question #1

Let **P** be the plane with equation $x + 2z = 0$. Find the distance from the point $(-1, 3, 0)$ to the plane **P**.

Example Solution #1

$$\textit{The plane passes through } (0,0,0) \textit{ and the normal vector } \vec{N} \textit{ is } \langle 1,0,2 \rangle$$

$$\textit{Create a vector } \vec{V} \textit{ from } (0,0,0) \textit{ to the point } (-1,3,0) \rightarrow \langle -1,3,0 \rangle$$

The magnitude of the projection of $\vec{\mathrm{V}}$ onto $\vec{\mathrm{N}}$ will be the distance from the point to the plane

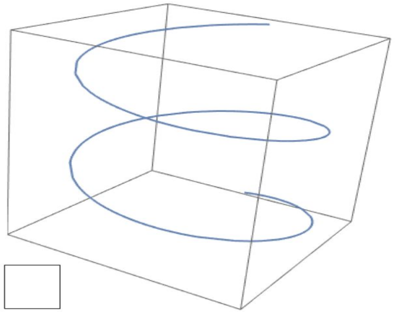
$$proj_{\vec{N}}\vec{V} = \frac{\vec{V}\cdot\vec{N}}{|\vec{N}|^2}\vec{N} = \langle {}^{-1}/_{5}, 0, {}^{-2}/_{5} \rangle$$

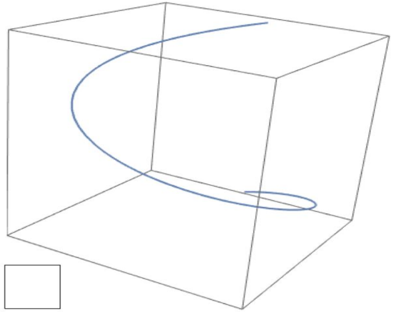
$$|proj_{\vec{N}}\vec{V}| = \sqrt{\left({}^{-1}/_{5}\right)^2 + \left({}^{-2}/_{5}\right)^2} = {}^{1}/_{\sqrt{5}}$$

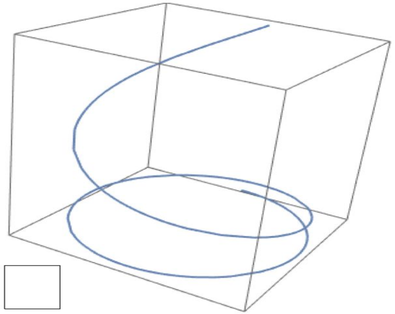
$$\textit{The distance is } 1/\sqrt{5}$$

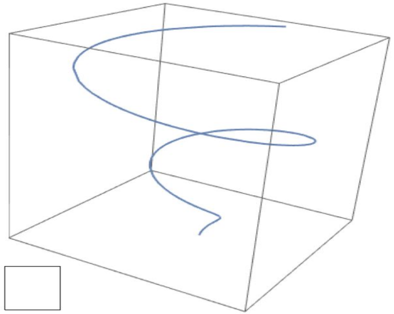
Example Question #2

Let C be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \leq t \leq 2\sqrt{\pi}$. Check the corresponding picture of C.



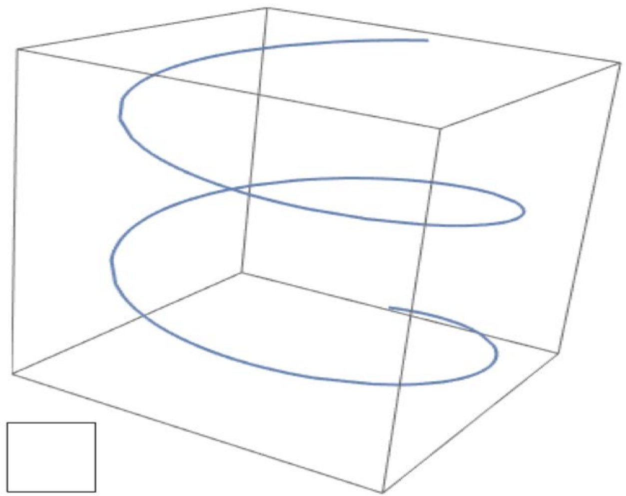






Example Solution #2

$\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \le t \le 2\sqrt{\pi}$



Example Question #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface $z = xy$.

Example Solution #3

Find the vector function representing the curve of intersection between the circular cylinder of radius 4 centered on the z-axis and the surface z = xy.

$$\overrightarrow{r_{cyl}} = \langle 4\text{cost}, 4\text{sint} \rangle$$

$$z = xy = 16cost \cdot sint$$

$$\vec{r}(t) = \langle 4\text{cost}, 4\text{sint}, 16\text{cost} \cdot \text{sint} \rangle$$

Limits

- When computing multivariable limits,
  - Check **multiple paths** (lines and power functions) to see if there are conflicting values. If so, limits DNE
  - **Factor** (difference of squares)
  - Use **polar coordinates**
  - Try **squeeze theorem**

$$r = \sqrt{x^2 + y^2}$$
$$x = r \cdot cos\theta$$
$$y = r \cdot sin\theta$$

Squeeze Theorem

- We have three functions such that near x: $f(x) \leq g(x) \leq h(x)$

- If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

- Great to use for functions that are hard to evaluate with limit laws

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Limit Laws

Sum Law	**1.** The limit of a sum is the sum of the limits.
Difference Law	**2.** The limit of a difference is the difference of the limits.
Constant Multiple Law	**3.** The limit of a constant times a function is the constant times the limit of the function.
Product Law	**4.** The limit of a product is the product of the limits.
Quotient Law	**5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

Continuity

- A function f(x,y) is continuous at point (x,y) if

$$lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

- If this holds for all points (a,b), then the function is continuous over the 2D plane.

Partial Derivatives

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$f(x,y) \qquad \Rightarrow \qquad f_x(x,y) = \frac{\partial f}{\partial x} \;\; \& \;\; f_y(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy}$$

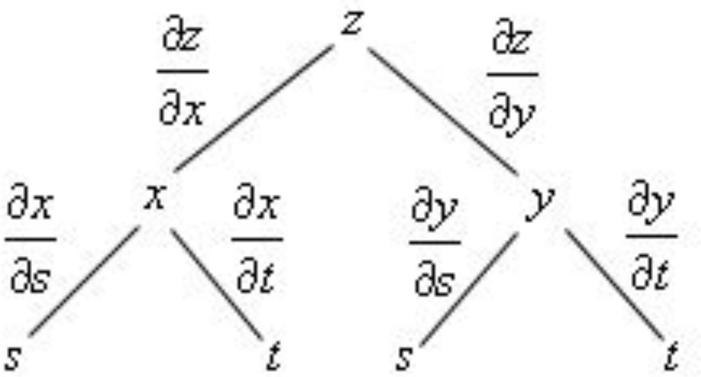
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

Arc Length Formula

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Chain Rule



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$\frac{dz}{dt}$: change in z with respect to t

$\frac{\partial z}{\partial x}$: z_x

$\frac{dx}{dt}$: change in x with respect to t

Linear Approximation & Tangent planes

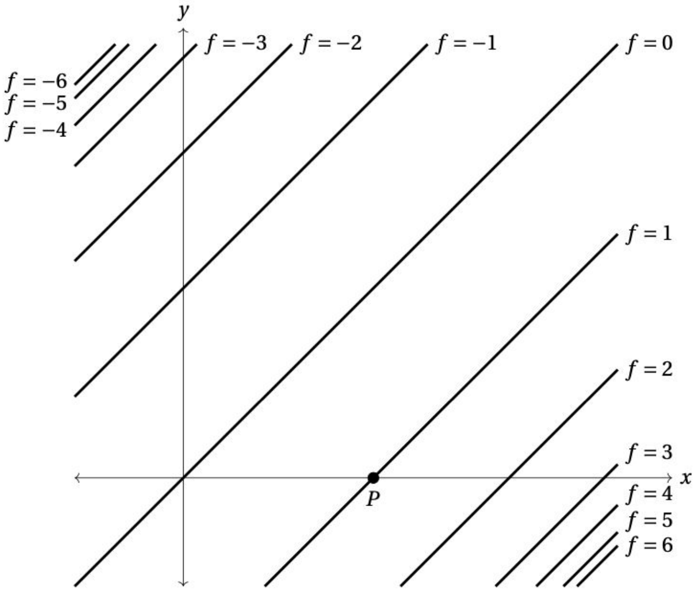
- If z = f(x, y) and f is **differentiable** at (a, b), then the value of f(m, n) can be approximated by

$$f(m, n) \approx L(m, n)$$

$$L(m, n) = f(a, b) + f_x(a, b) \cdot (m - a) + f_y(a, b) \cdot (n - b)$$

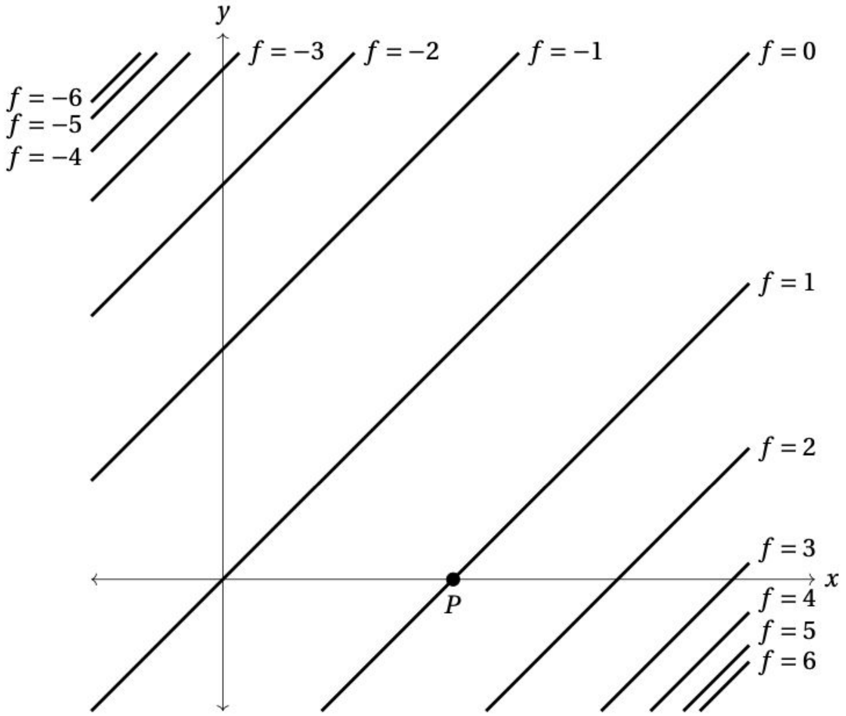
Example Question #4

- A contour map for a function f of x, y, and a point P in the plane are given below. Determine if the following quantities are negative, zero, or positive: $f_x(P)$, $f_{xx}(P)$, $f_{xy}(P)$



Example Solution #4

- $f_x(P)$: positive
- $f_{xx}(P)$: positive
- $f_{xy}(P)$: negative



Example Question #5

- Compute the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8} \qquad \lim_{x\to 0} x^3 cos\left(\frac{2}{x}\right) \qquad \lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$$

- Determine whether the following function is continuous at (0, 0)

$$f(x,y) = \begin{bmatrix} \dfrac{xy}{x^2+xy+y^2}, (x,y) \neq (0,0) \\ 0 \qquad\qquad\quad , (x,y) = (0,0) \end{bmatrix}$$

Example Solution #5

- Compute the following limits

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$$ **= 0** (Use polar coordinates)

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$$ **= DNE** (Check $x = y^4$ and $x = -y^4$)

$$\lim_{x\to 0} x^3 cos\left(\frac{2}{x}\right)$$ **= 0** (Squeeze Theorem)

$$\lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$$ **= 4** (Plug in (–1, 0) directly)

Example Solution #5

- Determine whether the following function is continuous at (0, 0)

$$f(x,y) = \begin{bmatrix} \dfrac{\mathrm{xy}}{\mathrm{x}^2 + \mathrm{xy} + \mathrm{y}^2}, (x,y) \neq (0,0) \\ 0 \qquad\qquad\qquad , (x,y) = (0,0) \end{bmatrix}$$

On line y = x, f(x, y) = 1/3 at any point except (0, 0). Since there is a discontinuity at (0, 0), the function is not continuous.

Velocity and Acceleration

$$\vec{r}'(t) = \vec{v}(t) \rightarrow \vec{r}(t) = \int \vec{v}(t)\, dt$$

$$\vec{v}'(t) = \vec{a}(t) \rightarrow \vec{v}(t) = \int \vec{a}(t)\, dt$$