Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Phys 213 – University Physics: Thermal Physics

Final Review Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Wednesday, December 10, 6-8 pm Luke, Zaahi, Camille

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1 Chemical Potential and Phases

1. Label each statement as True or False.

(a) Temperature is to heat as chemical potential is to the number of particles.

(b) Suppose we are in a constant pressure and temperature scenario, and we have a substance such that $\mu_{\text{gas}} < \mu_{\text{liquid}}$. After reaching equilibrium, the substance will completely be gaseous.

(c) In the situation in (b), μ_{gas} gets bigger when particles evaporate.

(d) By adding an ideal solute to water, we can lower both its melting point and boiling point.

(e) At a substance's melting point, $\mu_{\text{liquid}} = \mu_{\text{solid}}$.

(f) The temperature of a substance changes while it is undergoing a phase change.

(a) True. Heat transfers from high to low temperatures, and particles transfer from high to low chemical potentials.

(b) True. Particles favor lower chemical potentials, so they would rush to the gas state.

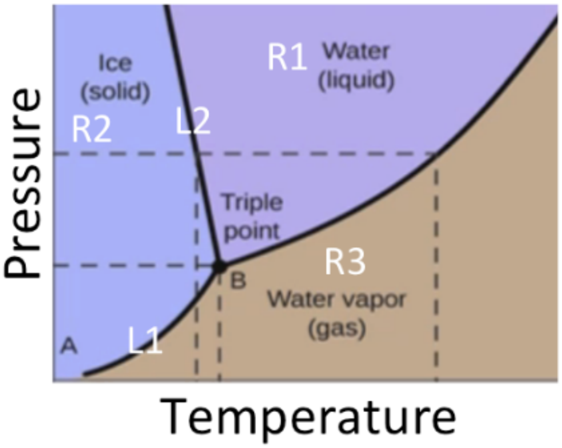
(c) False. In a constant temperature and pressure scenario, the chemical potentials are constant.

(d) False. Adding a solute would lower the melting point but raise the boiling point.

(e) True. At a phase transition point, the chemical potentials are equal, the substance can exist in a superposition of liquid and solid.

(f) False. During a phase change, a substance's temperature is constant.

2. Below is the phase diagram for water. Determine which chemical potential(s) is the lowest at the following points on the graph.



(a) region R1

(b) line L1

(c) extremely high temperatures, low pressures

(a) liquid
(b) solid and gas
(c) gas

3. Given a latent heat (vaporization or fusion), determine the heat energy required to perform the given phase change.

(a) heat of vaporization: 15 kJ/kg: boiling 20 kg of substance
(b) heat of fusion: 34 kJ/mol: melting 12 mol of substance
(c) heat of vaporization: 20 kJ/kg, boiling 15 mol of substance (molar mass: 40 g/mol)

(a) 300 kJ
(b) 408 kJ
(c) 12 kJ

4. Given a specific latent heat, a molar mass, and a phase transition temperature, determine the increase in entropy per molecule when the substance undergoes that phase change.

$$L = T\Delta S$$

(a) $L = 153$ kJ/kg, $M = 35$ g/mol, $T = 135$ K
(b) $L = 230$ kJ/kg, $M = 15$ g/mol, $T = 300$ K
(c) $L = 181$ kJ/kg, $M = 54$ g/mol, $T = 275$ K

(a) 6.587×10^{-23} J/K
(b) 1.910×10^{-23} J/K
(c) 5.902×10^{-23} J/K

5. We have a substance whose boiling point at 50 kPa is 230 K. Its latent heat per kilogram is 2500 J/kg, and its molar mass is 22 g/mol. Suppose we want to increase its boiling point by 0.5 K. This is small enough that we approximate $\mathrm{d}p \approx \Delta P$ and $\mathrm{d}T \approx \Delta T$. Assume the liquid phase of the substance has negligible volume.

(a) First, set up the differential relation between $\mathrm{d}\mu_L$ and $\mathrm{d}\mu_G$ at the boiling point (chemical potentials of the liquid and gas phases).
(b) Assume that the gas phase of this substance is ideal. Determine V_G/N_G, the volume per gas particle at the given pressure and temperature.
(c) Recall that for latent heat, $L = T\Delta S$. Use this to determine $\Delta S/N = \Delta S_G/N_G - \Delta S_L/N_L$, the change in entropy per particle when converting from liquid to gas.

(d) Use (a) - (c) to determine the change in pressure necessary to create the desired increase in boiling point temperature ($\Delta T = 0.5$ K).

(a) We set $\mu_L = \mu_G$, realizing that $V_L = 0$:

$$-\frac{S_L}{N_L}\mathrm{d}T = \frac{V_G}{N_G}\mathrm{d}p - \frac{S_G}{N_G}\mathrm{d}T$$

(b) Using ideal gas law:

$$\frac{V}{N} = \frac{kT}{p}$$

Plugging in the initial temperature and pressure yields 6.348×10^{-26} m^3.

(c) We are given the latent heat per kilogram, which we will label as L_B. If we want to convert this to the latent heat per particle, we must do some units conversions:

$$\frac{2500 \text{ J}}{1 \text{ kg}} \times \frac{0.022 \text{ kg}}{1 \text{ mol}} \times \frac{1 \text{ mol}}{6.022 \times 10^{23} \text{ particles}} = 9.133 \times 10^{-23} \text{ J/particle}$$

Then, since $L = T\Delta S$, if we divide the latent heat per particle by T, we will end up with the change in entropy per particle. Here, this ends up being 3.971×10^{-25} J/particle K.

(d) If we rearrange our equation in (a):

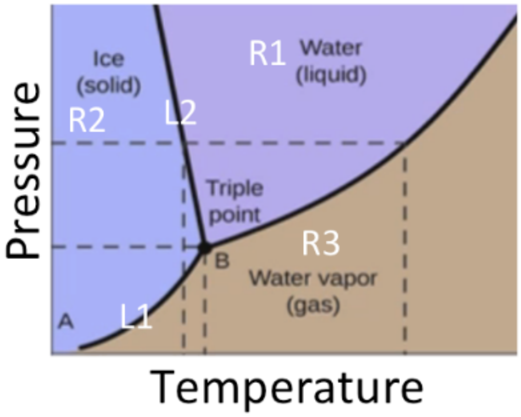
$$\frac{V_G}{N_G}\mathrm{d}p = \left(\frac{S_G}{N_G} - \frac{S_L}{N_L}\right)\mathrm{d}T$$

On the left, we have the volume per particle we found in (b), and on the right we have the entropy change per particle we found in (c). If we approximate the differentials to be “small changes“ and plug in 0.5 K for dT, along with the other values, we get that $\Delta p = 3.128$ Pa.

6. Why does adding salt on the sidewalks in winter prevent ice from forming?
When we add salt to a water-based solution, the entropy of the solution increases because there are more microstates associated with it. This increased entropy makes it harder for the substance to freeze, since nature favors higher entropy systems. That is, the temperature would have to go even lower than it currently is for nature to favor the solid form of this substance over its liquid form. As a result, the freezing point decreases.

An alternative way to answer this question is to remember that adding solute to a solution lowers the chemical potential of the liquid. Consequently, the melting point decreases and the boiling point increases. This means that the ice will melt at a cooler temperature; thus preventing ice from forming in the winter.

7. How does the phase diagram for water differ from other pure substances?
The phase diagram of water is unique in that as we increase pressure, the substance will become a liquid as opposed to remaining a solid. This indicates that the density of the solid form of water (ice) is less than the density of the liquid form, which is why ice floats in water.

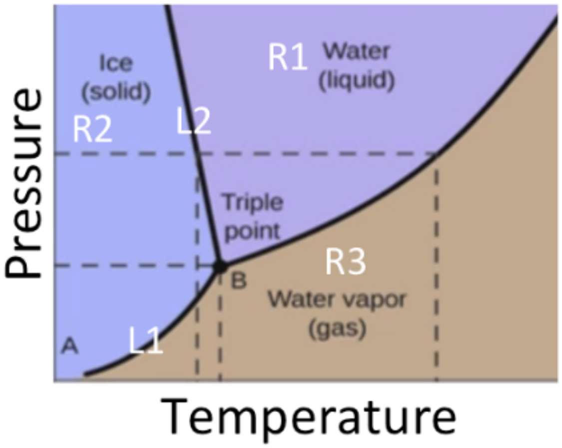


8. A substance has the following phase densities:

phase	ρ
solid	23 $\mathrm{kg/m^3}$
liquid	25 $\mathrm{kg/m^3}$
gas	1.3 $\mathrm{kg/m^3}$

At extremely highly pressures, which phase will have the lowest chemical potential?
We expect liquid to have the lowest chemical potential. At high pressures, the densest state of a substance is favored, which in this case is the liquid phase.

9. Here is the phase diagram for water.



(a) In which region or line is $\mu_{\text{liquid}} = \mu_{\text{solid}} < \mu_{\text{gas}}$?

(b) We have water exactly at its melting point (line L2). If we add an ideal solute to the water while keeping the temperature the same, which phase will the water become?

(a) Line L2 satisfies the condition. It is the melting point of water.

(b) Adding a solute to a liquid lowers its chemical potential curve. Because of this, we expect the melting point to lower, resulting in the water converting to the liquid phase.

10. A sample of Substance A is at its melting point T = 313.3 K. Given that it takes 1566.5 J of energy to melt the sample at this temperature, what is the change in entropy per particle of the sample, given that the sample is made up of 1.5 moles of substance A?

(a) 5.5×10^{-24} J/K/particle

(b) 3.33 J/K/particle

(c) 5 J/K/particle

(d) 7.624×10^{-24} J/K/particle

The latent heat of a substance is:

$$L = T\Delta S \tag{1}$$

We are given that it takes 1566.5 J of energy to melt the substance, this must be L since it represents the total amount of energy necessary to complete a phase change for our sample. Dividing by T, we get:

$$L/T = 1566.5/313.3 = 5\text{J/K}$$

However, we are asked for the entropy per particle. We first convert the moles of substance A to particles by multiplying 1.5 by Avogadro's number, giving us the total number of particles. Then, we divide our total entropy change by this value to get:

$$\frac{5}{1.5 * N_A} = 5.5 \times 10^{-24}\text{J/K per particle} = \Delta S$$

2 The Boltzmann Factor

1. Label the following statements as true or false.

(a) A quantum harmonic oscillator can take on any value of energy.

(b) The total energy of an array of quantum harmonic oscillators is constant, but the individual energies may change.

(c) The Boltzmann factors of each microstate of a system always sum to 1.

(d) As temperature goes to 0, the Boltzmann factor approaches 0 and each microstate of a system approaches an equal probability.

(e) As temperature goes to infinity, each microstate of a system approaches an equal probability.

(a) False. A quantum harmonic oscillator can only take on discrete, quantized amounts of energy.

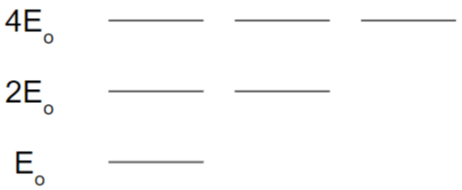
(b) True. As long as energy conservation is conserved, the quantum harmonic oscillators can take on different individual energies.

(c) False. This is not guaranteed, and normalization must be done to get the correct probabilities.

(d) False. While it is true that the Boltzmann factor approaches 0 as temperature goes to 0, this does not mean that each microstate is equally likely. Instead, the ground state becomes the most likely state.

(e) True. Since the Boltzmann factor for each microstate approaches 1, they each take on an equal probability.

2. Suppose we have a particle in contact with a reservoir at temperature T with the following microstates available to it:



Notice that certain energy levels have more than one valid microstate. At $T = E_\circ/k$:

(a) Determine the probability of the particle having energy $4E_\circ$.

(b) Determine the average energy of the particle in terms of $E_\circ$.

(a) To find the probability of a given energy level, we must find the net probability of getting any microstate associated with that energy. Here, $4E_\circ$ has 3 available microstates, so its individual Boltzmann factor is

$$3e^{-4E_\circ/kT} = 3e^{-4}$$

We can do some cancelling since $T = E_\circ/k$. To find the probability, we must normalize the Boltzmann factor by dividing by the sum of all of the Boltzmann factors:

$$\begin{aligned} P(4E_\circ) &= \frac{3e^{-4}}{e^{-1} + 2e^{-2} + 3e^{-4}} \\ &= 0.0792 \end{aligned}$$

(b) The average energy is just the weighted average of each energy:

$$\begin{aligned} E_{\text{avg}} &= E_\circ P(E_\circ) + 2E_\circ P(2E_\circ) + 4E_\circ P(4E_\circ) \\ &= \frac{E_\circ(e^{-1}) + 2E_\circ(2e^{-2}) + 4E_\circ(3e^{-4})}{e^{-1} + 2e^{-2} + 3e^{-4}} \\ &= 1.628E_\circ \end{aligned}$$

3. We have a system with a single quantum harmonic oscillator. In this system, one energy quantum is $\varepsilon = 1.5 \times 10^{-21}$ J, and there are a total of $q = 4\varepsilon$ energy quanta available. The oscillator is in a $T = 350$ K environment.

(a) Determine the ratio of probabilities for observing the $E = 4\varepsilon$ energy state and $E = 3\varepsilon$ energy state.

(b) As we increase the temperature T, what happens to the probabilities of each state?

(a) To find the ratio of probabilities between two energies, we simply divide their Boltzmann factors:

$$\begin{aligned} \frac{P(4\varepsilon)}{P(3\varepsilon)} &= \frac{e^{-4\varepsilon/kT}}{e^{-3\varepsilon/kT}} \\ &= e^{-\varepsilon/kT} \\ &= 0.733 \end{aligned}$$

(b) The states begin to take on the same probability. Initially, the lowest energy state ($E = 0$) is the most likely, but the probability of each state gradually becomes equally likely. This is because as temperature T increases, the Boltzmann factor approaches 1, regardless of the energy.

4. We have a particle with two possible energy states $\pm\varepsilon$, where $\varepsilon = 6 \times 10^{-22}$ J. It is in contact with a thermal reservoir at temperature T.

(a) Determine the temperature T such that $P(-\varepsilon)/P(+\varepsilon) = 20$.

(b) What is the average energy at this temperature?

(c) What is the probability of measuring $+\varepsilon$ as T becomes extremely large? What is the average energy at extremely large temperatures?

(a) We want $P(-\varepsilon)/P(+\varepsilon)$ to be 20, or

$$\begin{aligned}\frac{e^{-(-\varepsilon)/kT}}{e^{-\varepsilon/kT}} &= 20\\ e^{2\varepsilon/kT} &= 20\end{aligned}$$

Taking the natural log of both sides:

$$\begin{aligned}\frac{2\varepsilon}{kT} &= \ln(20)\\ T &= \frac{2\varepsilon}{k\ln(20)}\\ &= 28.027\text{ K}\end{aligned}$$

(b)

$$\begin{aligned}E_{\text{avg}} &= (-\varepsilon)P(-\varepsilon) + (+\varepsilon)P(+\varepsilon)\\ &= \frac{-\varepsilon e^{\varepsilon/kT} + \varepsilon e^{-\varepsilon/kT}}{e^{\varepsilon/kT} + e^{-\varepsilon/kT}}\\ &= E_{\text{avg}} = -5.429\times 10^{-22}\text{ J}\end{aligned}$$

(c) In the limit as T goes to infinity, the Boltzmann factor goes to 1, regardless of energy. So

$$\lim_{T\to\infty} P(+\varepsilon) = \frac{1}{1+1} = \frac{1}{2}$$

$P(-\varepsilon)$ will also be 1/2 — each state will become equally likely. Thus the average energy is

$$E_{\text{avg}} = -\varepsilon(1/2) + \varepsilon(1/2) = 0\text{ J}$$

5. An astrophysics major is trying to measure the temperature of a distant star. The star is primarily comprised of hydrogen. For simplicity, we'll describe the energy states of hydrogen as the following:

Here, $E_0 = 0$ J and $E_1 = 1.634\times 10^{-18}$ J. Through science-y magic, the student determines that 99.9999993% of the hydrogen in the star is in the E_0 energy state.

(a) Calculate the ratio of hydrogen atoms in the E_0 state to those in the E_1 state.

(b) Determine the temperature. Remember that the energy E_1 has 4 available microstates.

(a) If 99.9999993% of the hydrogen atoms are in the $E_\circ$ state, this means that $100\% - 99.9999993\% = 7\times 10^{-7}\%$ are in the E_1 state.

$$\begin{aligned}\frac{N_0}{N_1} &= \frac{99.9999993\%}{7\times 10^{-7}\%}\\ &= 1.429\times 10^{8}\end{aligned}$$

(b) The ratio of hydrogen atoms in the E_0 state versus the E_1 state should be indicative of their probabilities:

$$\frac{N_0}{N_1} = \frac{P(E_0)}{P(E_1)}$$

To find this ratio, we divide the Boltzmann factors:

$$\begin{aligned}\frac{P(E_0)}{P(E_1)} &= \frac{e^{-E_0/kT}}{4e^{-E_1/kT}} \\ &= \frac{1}{4}e^{-(E_0-E_1)/kT}\end{aligned}$$

This should be equal to the ratio we found above:

$$\begin{aligned}\frac{1}{4}e^{-(E_0-E_1)/kT} &= 1.429 \times 10^8 \\ \frac{-(E_0-E_1)}{kT} &= \ln\big(4(1.429 \times 10^8)\big) \\ T &= \frac{E_1 - E_0}{k\ln(4(1.429 \times 10^8))} \\ &= 5872 \text{ K}\end{aligned}$$

6. We have a 10 km-tall cylinder filled with helium particles. The cylinder is heated to $T = 500$ K. The mass of a helium particle is $m_{\text{He}} = 6.643 \times 10^{-24}$ kg.

(a) Write the potential energy of a single Helium particle as a function of height h.

(b) Determine the ratio of probabilities between a particle at height h_1 and a particle at height h_2 as a function of h_1 and h_2.

(c) Locations of high probability in the cylinder correspond to higher-pressure areas, i.e.

$$p \propto P(h)$$

Determine the ratio of pressures between the top and bottom of the cylinder.

(a) This is simply the classical gravitational potential energy:

$$U(h) = m_{\text{He}}gh$$

(b) To find the ratio of probabilities, we divide the Boltzmann factors. The particle at h_1 has energy $m_{\text{He}}gh_1$, the one at h_2, $m_{\text{He}}gh_2$:

$$\frac{P(h_1)}{P(h_2)} = \frac{e^{-m_{\text{He}}gh_1/kT}}{e^{-m_{\text{He}}gh_2/kT}} = e^{-m_{\text{He}}g(h_1-h_2)/kT} \tag{2}$$

(c) Pressure is proportional to probability, so

$$\frac{p_1}{p_2} = \frac{P(h_2)}{P(h_1)}$$

which is just the ratio we wrote in (b). The difference in heights $h_1 - h_2$ is 10 km (the height of the cylinder), so

$$\frac{p_{\text{top}}}{p_{\text{bottom}}} = e^{-m_{\text{He}}g(h_1-h_2)/kT} = 1.0577 \times 10^{-41}$$

7. This question is pretty math/proof heavy and thus won't be asked on a test.
   Let's create a formula for determining the heat capacity of a two-state system, $C(T)$.

(a) Suppose the two-state system has energies E_1 and E_2. Write the Boltzmann factor for each state.

(b) Write an expression for the average energy. This is the internal energy as a function of temperature T, $U(T)$.

(c) Find the heat capacity $C(T)$ by taking the derivative of $U(T)$ with respect to temperature (since $C = \mathrm{d}U/\mathrm{d}T$). In case you forgot your derivative rules:

$$x\left(\frac{f}{g}\right) = \frac{f'g - g'f}{g^2}$$

You can also use an online derivative calculator ; this isn't a calculus class after all.

(d) Use this formula to find the heat capacity for a system with energy states $E_1 = 0$ J and $E_2 = 5.5 \times 10^{-20}$ J at temperature $T = 400$ K.

(a) $E_1 : \ e^{-E_1/kT}$, $E_2 : \ e^{-E_2/kT}$

(b) The average energy is the weighted average:

$$\begin{aligned} U(T) &= E_1 P(E_1) + E_2 P(E_2) \\ &= \frac{E_1 e^{-E_1/kT} + E_2 e^{-E_2/kT}}{e^{-E_1/kT} + e^{-E_2/kT}} \end{aligned}$$

(c) After miraculously finding the derivative:

$$\frac{dU}{dT} = C(T) = \frac{(E_1 - E_2)^2 e^{(E_1+E_2)/kT}}{kT^2(e^{E_1/kT} + e^{E_2/kT})^2}$$

(d) $C(400 \text{ K}) = 6.449 \times 10^{-26}$ J/K

8. We are given a quantum system with 3 quantum harmonic oscillators and 7 Quanta of energy to distribute into them. How many different ways can we distribute the energy into the system?
   From our formula sheet, we can use the binomial coefficient expansion formula.

$$\binom{N-1+q}{q} = \binom{3-1+7}{7} = \frac{9!}{7!2!} = 36$$

So there are 36 different ways to distribute the energy into the oscillators.

9. Let's say we have an array of 8 quantum harmonic oscillators with $q = 20$ total energy quanta.

(a) How many microstates does this system have? What is the system's entropy?

(b) Now let's say we fix one of the oscillators to have 4 energy quanta (i.e. $E = 4\varepsilon$). How many microstates does *this* system have? What is its entropy?

(c) Without doing any calculations, if we were to fix one oscillator to 0 energy quanta instead of 4, would we expect the resulting entropy to be higher or lower?

(a) We use the arrangement formula for QHOs:

$$\binom{N+q-1}{q} = \binom{8+20-1}{20} = \binom{27}{20} = 888,030$$

So there are $\Omega = 888,030$ possible microstates. To find the entropy, we plug this into $S = k\ln(\Omega)$ to get 1.89×10^{-22} J/K.

(b) If we fix one oscillator to have 4 energy quanta, then we have 7 remaining oscillators and 16 remaining energy quanta that we can arrange.

$$\binom{7+16-1}{16} = 74,613$$

So $\Omega = 74,613$, and the entropy is $S = 1.548 \times 10^{-22}$ J/K.

(c) We expect it to be higher, as we will have 7 QHOS with more energy quanta available to the rest of them.

3 Past Units

1. Consider 5 coins, each initially starting on heads.

(a) What is the entropy, S, of this system in its current configuration?

(b) List all the macrostates available to this system.

(c) Identify the most probable macrostates. Hint: there are two.

(d) How many microstates would lead to the macrostates identified above?

(e) Calculate the change in entropy, ΔS, if the system changed to either of its most probable macrostates.

(a) The first thing to note here is that this system is in a 5H 0T *macrostate* (in this scenario, macrostates are the number of heads up (or tails) in a given configuration). There's only one way to attain 5 heads — all of them are heads! So there is one *microstate* corresponding to this macrostate. We can now use the statistical definition of entropy, $S = k_b \ln(\Omega)$, with the microstates in this configuration, Ω, equalling one. Thus,

$$\boxed{S = k_b \ln(1) = 0} \tag{3}$$

Which is a reasonable answer because this system is highly uniform.

(b) The macrostates of this system are all the combinations of heads (H) and tails (T) available to the 5 coins.

5H 0T
4H 1T
3H 2T
2H 3T
1H 4T
0H 5T

(c) The most probable macrostates have the highest number of associated microstates.

3H 2T
2H 3T

(d) To calculate the number of microstates in either of these macrostates, we must use the combination formula, ${}_nC_r$, where n is the total number of items in the set (5), and r is the number of items we're choosing (hence *n choose r*).

$$\boxed{{}_5C_3 =_5 C_2 = 10 \text{ microstates}}$$

(e) The change in entropy is found by taking the difference of the final state entropy, S_{final}, and initial state entropy, $S_{initial}$.

$$\Delta S = S_{final} - S_{initial} = k_b \ln(10) - k_b \ln(1) = \boxed{k_b \ln(10)}$$

Note that the entropy increased, as we would expect by the second law of thermodynamics.

2. Let's investigate the classic "cook a whole chicken by slapping it" experiment. The average whole raw chicken has a mass of 1.4 kg with a specific heat capacity of 3350 J/kg K. Your hand (along with a heat-insulating glove you're wearing) weighs 0.7 kg. Let's assume the chicken is in an insulated environment and is held in place, so the chicken cannot transfer heat to its surroundings and cannot move. The chicken starts out at room temperature, $T_\circ = 298.15$ K.

(a) Determine how much energy must be added to the chicken to fully cook it, i.e. bring it to $T_f = 350$ K.

(b) If you wanted to cook the chicken in one slap, determine how fast your hand must be moving during the slap.

(c) If you wanted to cook the chicken with multiple normal slaps (≈ 7 m/s), determine how many slaps you would need.

(a)

$$\begin{aligned} U &= C\Delta T \\ &= (1.4)(3350)(350 - 298.15) \text{ J} \\ &= 243.177 \text{ kJ} \end{aligned}$$

(b) We want the kinetic energy of the slap to be equivalent to energy needed to cook the chicken.

$$\begin{aligned} \frac{1}{2}mv^2 &= 243177 \text{ J} \\ v &= \sqrt{\frac{(2)(243177)}{0.7}} \text{ m/s} \\ &= 833.541 \text{ m/s} \end{aligned}$$

For reference, the speed of sound in air is 343 m/s.

(c) We'll need N normal slaps of kinetic energy K:

$$\begin{aligned} (N)\frac{1}{2}mv^2 &= 243177 \text{ J} \\ N &= \frac{(2)(243177)}{(0.7)(7)^2} \\ &= 14179.417 \end{aligned}$$

So you'd need at least 14180 normal slaps to cook the chicken.

3. Two blocks, A and B, come in to contact. Block A starts out at $T_A = 150$ K, while block B starts at $T_B = 400$ K. The heat capacity of block A is 15 J/K, and that of block B is 5 J/K.

(a) Suppose Block A has a mass of 5 kg, for Block B, 1 kg. What would be the *specific* heat capacity for each?

(b) Determine the equilibrium temperature, T_f.

(c) Determine the net change in entropy. Which block lowered in entropy, and which block rose in entropy? Hint:

$$\frac{1}{T} = \frac{\partial S}{\partial U}, \; C = \frac{\partial U}{\partial T}$$

(a) Remember that $C = mc$, where C is the heat capacity and c is the specific heat capacity. Thus $c_A = 3$ J/kg K and $c_B = 5$ J/kg K.

(b) We can use the weighted average formula to do this quickly:

$$\begin{aligned} T_f &= \frac{(15)(150) + (5)(400)}{15 + 5} \text{ K} \\ &= 212.5 \text{ K} \end{aligned}$$

(c) To find the change of entropy, we perform the integral

$$\begin{aligned} \Delta S &= \int_{U_i}^{U_f} \frac{\mathrm{d}U}{T} \\ &= \int_{T_i}^{T_f} \frac{C\,\mathrm{d}T}{T} \end{aligned}$$

for each block.

$$\begin{aligned} \Delta S_A &= \int_{150}^{212.5} \frac{(15)\,\mathrm{d}T}{T} \text{ J/K} \\ &= 15 \ln(212.5/150) \text{ J/K} \\ &= 5.225 \text{ J/K} \\ \Delta S_B &= 5 \ln(212.5/400) \text{ J/K} \\ &= -3.163 \text{ J/K} \end{aligned}$$

Thus block A increases in entropy while block B decreases in entropy. The net entropy change is

$$\begin{aligned} \Delta S &= 5.225 \text{ J/K} - 3.163 \text{ J/K} \\ &= 2.062 \text{ J/K} \end{aligned}$$

which, as expected, is positive.

4. Timmy buys an ice cream cone on a hot summer day, but he gets distracted and leaves it on a park bench. The specific latent heat of fusion of ice cream is 2.34×10^5 J/kg, and his scoop has a mass of 75 g.

(a) If the sun is adding energy to his ice cream at a rate of 5 W, estimate how long it will take for his ice cream to completely melt, assuming it's already at its melting point.

(b) Now, let's say the specific heat capacity of melted ice cream is 2400 J/kg K. Assuming that all of the ice cream has to melt before the liquid ice cream starts to increase in temperature, and that the melting point of ice cream is about 273.15 K (which is also its initial temperature), determine the total time for the ice cream to go from solid to a room temperature liquid (room temp. = 298.15 K).

(a) The amount of energy needed to melt the ice cream is simply $(2.34 \times 10^5)(0.075)$ J = 17550 J. The rate of energy being added to the scoop is 5 J/s, so the time t required to melt the scoop is 17550/5 s = 3500 s, a little less than an hour.

(b) We already found the time needed to melt the ice cream, so all we need is the time to heat up the melted ice cream. The energy needed to get the ice cream to room temperature is $(2400)(0.075)(298.15 - 273.15)$ J = 4500 J. The time it takes to do this is 4500/5 s = 900 s. Thus the entire melting and warming process takes $3500 + 900 = 4400$ s.

5. Determine the specific heat capacity of solid aluminum via equipartition. Use the value of molar mass in the equation sheet.

Solid aluminum has 6 DOFs, so

$$U = \frac{6}{2}NkT$$
$$C = \frac{\mathrm{d}U}{\mathrm{d}T} = 3Nk$$

for N particles of aluminum. There are N_A particles in a mole (Avogadro's number), so

$$c_m = 3N_Ak$$

is the molar heat capacity. Finally, using the molar mass of aluminum (0.028 kg),

$$c = \frac{3N_Ak}{M} = 890.4 \text{ J/ kg K}$$

6. We have helium gas at temperature of 400 K. The molar mass of helium is 4.003 g/mol.

(a) Determine the RMS velocity of the helium particles.

(b) How does the RMS velocity of these particles compare to the RMS velocity of neon gas at the same temperature? The molar mass of neon is 20.180 g/mol.

(a) The internal energy of a mole of helium, being a monatomic gas:

$$U = \frac{3}{2}N_AkT$$

This internal energy should correspond to the kinetic energy of the mole of helium. We are given the molar mass of helium, M:

$$\begin{aligned} m &= \frac{M}{N_A} && \text{(mass of one particle)} \\ &= \frac{4.003 \times 10^{-3}\ \text{kg/mol}}{6.022 \times 10^{23}\ \text{mol}^{-1}} \\ &\approx 6.64 \times 10^{-27}\ \text{kg} \end{aligned}$$

$$\begin{aligned} \frac{1}{2} m v_{\text{rms}}^2 &= \frac{3}{2} kT \\ v_{\text{rms}} &= \sqrt{\frac{3kT}{m}} \\ &\approx 1.58 \times 10^3\ \text{m/s}. \end{aligned}$$

(b) We would expect the neon particles to have a smaller RMS velocity, as it has a higher molar mass.

7. What is the relationship between volume and pressure during isothermal and adiabatic processes for an ideal gas, respectively?
In an isothermal process, the temperature is constant and we can apply the ideal gas law: $pV = Nk_bT = \text{constant}$ (The ideal gas law is also used for isobaric and isochoric processes).

In an adiabatic process, $pV^\gamma = \text{constant}$ where $\gamma = \frac{C_p}{C_v} = (D+2)/D$. Where C_v is the constant volume heat capacity, C_p is the constant pressure heat capacity, and D is the number of degrees of freedom. Since no heat is added (adiabatic) but work can still be done, $\Delta S = 0$. As the gas compresses, the temperature increases, and as the gas expands, the temperature decreases.

8. The following two questions refer to the setup described below.

A piston of volume 0.05 m^3 contains 5 moles of a monatomic ideal gas at 300 K. If it undergoes an isothermal process and expands until the internal pressure matches the external pressure, $P_E = 1$ atm.

(i) How much work is done by the gas on the environment?
   - a) 7.42×10^3
   - b) 1.12×10^4
   - c) -1.12×10^4
   - d) 1.83×10^4
   - e) -1.83×10^4

(ii) Suppose that the piston undergoes an adiabatic expansion instead, what is the final volume of the piston, V_f? (Values have units of cubic meters)
   - a) 0.086
   - b) 0.095
   - c) 0.123

(i) The answer is **(b)**. The work done ON the gas is $\mathrm{d}W_{on} = -p\mathrm{d}V$ while work done BY the gas is $\mathrm{d}W_{by} = p\mathrm{d}V$. They are different by a minus sign, so make sure to check your signs. To find the total work, we integrate both sides of the equation, writing p in terms of V using the ideal gas law

$$W_{by} = \int_{V_i}^{V_f} p\ dV = \int_{V_i}^{V_f} \frac{Nk_bT}{V}\ dV = Nk_bT\ln\left(\frac{V_f}{V_i}\right)$$

The final volume can be found using the ideal gas law (make sure pressure is in Pascals)

$$V_f = \frac{Nk_bT}{P_E} = 0.123$$

(ii) The answer is **(a)**. The P-V relation in an adiabatic process follows $pV^\gamma =$ constant. Since the ideal gas is monatomic, $\gamma = \frac{5}{3}$. We can find the initial pressure using ideal gas law

$$P_i = \frac{Nk_bT}{V_i} = 249420\ \mathrm{Pa}$$

which allows us to calculate the constant $PV^\gamma = 249420 \times 0.05^{\frac{5}{3}} = 1693$

After the expansion, we can use the $p - V$ relation again and plug in $P_f = P_E$ to get

$$P_EV_f^\gamma = 1693 \rightarrow V_f = 0.086$$

9. When a system is colder than the temperature of the environment (i.e. $\mathrm{T}_{sys} < \mathrm{T}_{eniv}$) its free energy is:

a) Smaller than its value when $T_{sys} = T_{eniv}$
b) Larger than its value when $T_{sys} = T_{eniv}$
c) The same as its value when $T_{sys} = T_{eniv}$

The answer is **(b)**. When the system reaches thermal equilibrium ($T_{sys} = T_{eniv}$), the free energy is minimized. So when the system has a different temperature from the environment, the free energy will always be larger than the minimum at equilibrium.

10. Using the second law of thermodynamics, show that it is impossible for a heat engine to operate at $\epsilon = 1$.

Start with the fact that $\Delta S_{total} = \Delta S_H + \Delta S_C$ (the total change in entropy is the sum of the change in entropy of the hot and cold reservoir).

We know from the definition of temperature that $\frac{1}{T} = \frac{\mathrm{d}S}{\mathrm{d}U}$. This let's us rewrite the above as

$$\Delta S_{total} = \frac{\Delta U_H}{T_H} + \frac{\Delta U_C}{T_C} = -\frac{Q_H}{T_H} + \frac{Q_C}{T_C}$$

Where the last step is a result of $Q = \Delta U$ since the reservoirs do no work. Now using the second law, $\Delta S_{total} \geq 0$, we can see $\frac{Q_C}{Q_H} \geq \frac{T_C}{T_H}$

Combining this with the definition of engine efficiency: $\epsilon = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}$
We can see the that there is a limit on ϵ

$$\boxed{\epsilon \leq 1 - \frac{T_C}{T_H}}$$