MATH 257 Final CARE Review

Please join the queue!

https://queue.illinois.edu/q/queue/955

We cannot start until everyone has joined!

Midterm 1 Topics

- Linear systems
    - Solving systems with matrices
- Reduced row echelon form
    - Pivot columns: basic and free variables
    - Row operations
- Vectors and spans
- Matrix operations
    - Addition, subtraction, scalar multiplication, linear combinations
    - Transposition
- Matrix multiplication
    - Properties of matrix multiplication
- Matrix inverses
    - What matrices are invertible?
    - Elementary matrices

Midterm 2 Topic Summary

- LU Decomposition
  - Lower/Upper Triangular Matrix
  - LU for Linear Systems
  - Permutation Matrix
- Vectors and Spans
  - Inner Product
  - Orthogonality
  - Linear Independence
- Subspaces
  - Column Space
  - Null Space
- Basis and Dimension
  - Fundamental Subspaces
  - Orthonormal bases
  - Orthogonal/normal Complements
- Graph and Adjacency Matrices
- Coordinates
  - Coordinate Matrices

Midterm 3 Topic Summary

- Linear Transformation
- Coordinate Matrices
- Determinants
- Eigenvectors and eigenvalues
- Markov Matrices
- Diagonalization
- Matrix powers
  - Matrix exponential
- Linear differential equations
- Matrix projections
- Least squares solutions

Topic Summary – New Content

- Gram-Schmidt Method
- Spectral Theorem
- SVD
- Low Rank SVD
- Pseudo Inverse
- PCA
- Complex Numbers

Elementary Matrices

Identity Matrices

$1 \times 1 \quad [1]$

$$2 \times 2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$3 \times 3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

etc.

Any matrix that can be form from the identity matrix with **one** elementary row operation.

Ex.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \to 3R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Matrix Inverses

Determinants:

$$\frac{1}{ad - bc}$$

For the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Definition of an inverse:

$$AC = I_n$$

Requirements for a matrix to be invertible:

1. It has to be square
2. The determinant of the matrix cannot be 0 or
3. The RREF of A is the identity matrix or
4. A has as many pivots as columns/rows

Statements 2, 3, and 4 mean the same thing.

Calculating an Inverse

For 2x2:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Elementary Matrix strategy:

$$A^{-1} = E_m E_{m-1} \ldots E_1 = E_m E_{m-1} \ldots E_1 I_n.$$

OR: set up an augmented matrix with the identity and reduce to RREF

Properties of Matrix Inverses

(a) A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$ *(i.e. A is the inverse of* A^{-1}*).*

(b) AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

(c) A^T *is invertible and* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Inverses are unique! Every invertible matrix only has one inverse.

Multiplying by a matrix inverse is the closest we get to dividing matrices.

Theorem 14. *Let A be an invertible* $n \times n$ *matrix. Then for each* $\mathbf{b}$ *in* $\mathbb{R}^n$*, the equation* $A\mathbf{x} = \mathbf{b}$ *has the unique solution* $\mathbf{x} = A^{-1}\mathbf{b}$.

Upper/Lower Triangular Matrices

Upper Triangular:

$$\begin{bmatrix} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Finding this is like doing REF with only row replacement

Lower Triangular:

$$\begin{bmatrix} \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 \\ \star & \star & \star & \ddots & \vdots \\ \star & \star & \star & \star & \star \end{bmatrix}$$

Keep track of your row operations to find L

LU Decomposition:

A = LU

- Not all matrices have LU decompositions
- LU decompositions are not unique (unlike inverses)

Finding the LU Decomposition

Determine the LU-decomposition of $\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}$

1) Col 1 Row 2 2) Col 1 Row 3 3) Col 3 Row 3

$$\begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix} \underset{R_2 \to R_2 \boxed{-4} R_1, R_3 \to R_3 \boxed{-4} R_1}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & -4 & 0 \end{bmatrix} \underset{R_3 \to R_3 \boxed{-1} R_2}{\rightsquigarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$, L := \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \qquad U := \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

LU for Linear Systems

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}}_{\mathbf{b}}$$

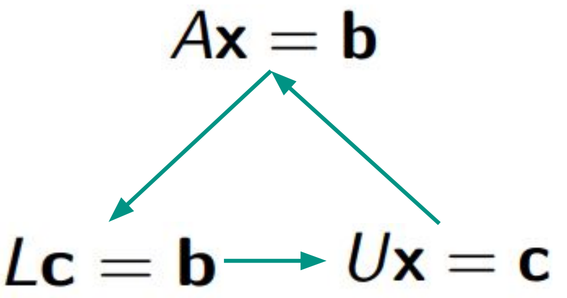
$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}}_{\mathbf{c}}.$$

Use **LU decomposition** to solve a linear system if:

1. A is nxn matrix
2. A = LU
3. $b \in R^n$

Step-by-step Algorithm

1. Find L and U
2. Solve for c using Lc = b
3. Solve for x using Ux = c



$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = L\mathbf{c} = \mathbf{b}$$

Permutation Matrices: for matrices that don't have an LU decomposition

Theorem 21. *Let A be $n \times n$ matrix. Then there is a permutation matrix P such that PA has an LU-decomposition.*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\rightsquigarrow} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = PA$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Step-by-step:

- Use the interchange operation done on **A** to an equivalent size identity matrix, this will be your **P** matrix
- Solve the for the **LU** decomposition of **PA**

When we apply the $\mathbf{P}^{-1}$ to **LU** (on the right), we'll be able to get the original value of **A**

$$PA = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ .5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:L} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=:U}$$

You Try: Solving with LU Decomposition

$$Given : A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$Solve \;\; A\vec{x} = \vec{b} \quad where \;\; b = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

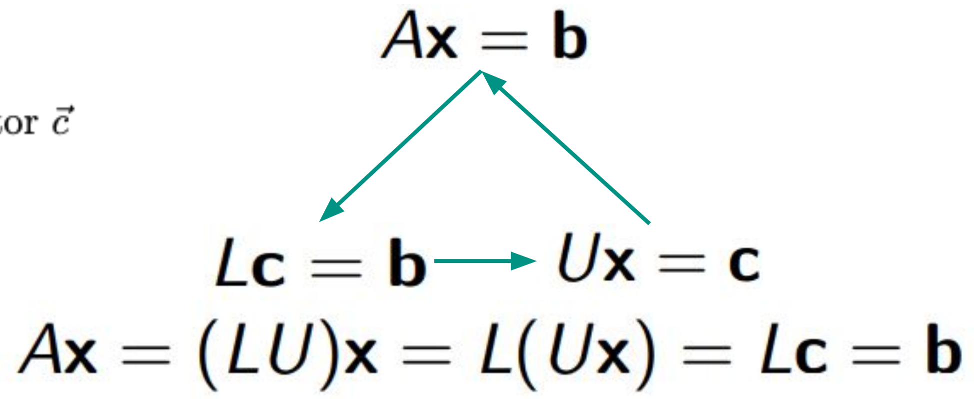
Do not compute A

provide values of $\vec{x}$ and the intermediate vector $\vec{c}$

For your reference:

Step-by-step Algorithm

1. Find L and U
2. Solve for c using Lc = b
3. Solve for x using Ux = c



Solution: Solving with LU Decomposition

Begin $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$I \quad Substitute\ U\vec{x} = \vec{c}$

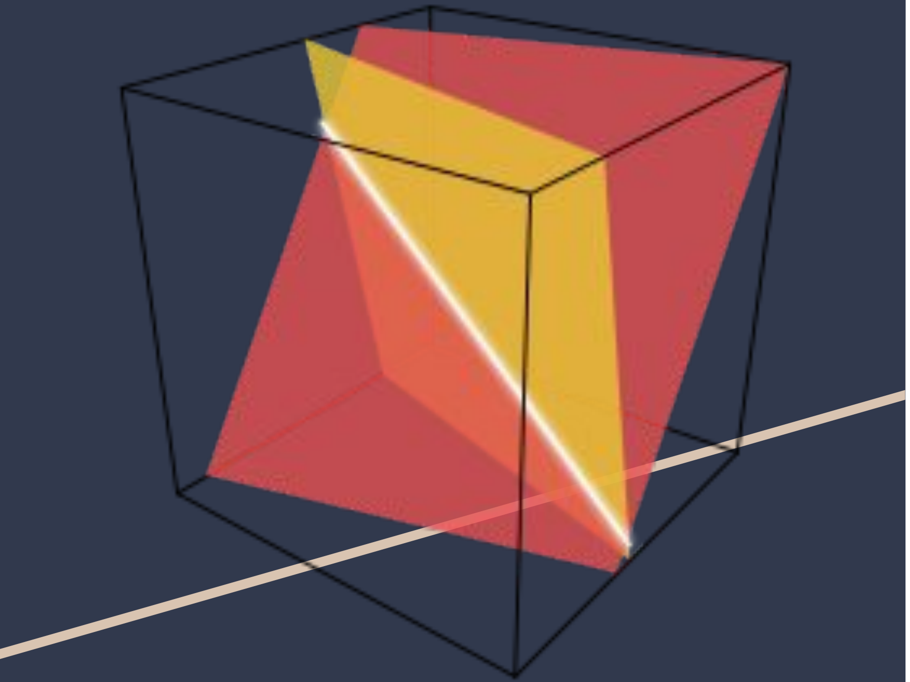
$II \quad Solve\ L\vec{c} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \xrightarrow{\text{fwd. sub.}} \vec{c} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$III \quad Solve\ U\vec{x} = \vec{c}$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \xrightarrow{\text{back sub.}} \vec{x} = \begin{bmatrix} -1 \\ -6 \\ -2 \end{bmatrix}$$

Subspaces



W is a **subspace** of **V,** if:

- W contains the 0 vector
- Adding any 2 vectors in W together gives a vector also in W
- Multiplying any vector in W by any scalar gives a vector also in W

Theorem 24. *Let* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in \mathbb{R}^n$. *Then* $Span(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m})$ *is a subspace of* $\mathbb{R}^n$.

Vector Spaces ‘V’

$u, v, w \in V$ and for all scalars $c, d \in \mathbb{R}$:

- $\mathbf{u} + \mathbf{v}$ is in V. (V is “closed under addition”.)
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- There is a vector (called the zero vector) $\mathbf{0}_V$ in V such that $\mathbf{u} + \mathbf{0}_V = \mathbf{u}$.
- For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}_V$.
- $c\mathbf{u}$ is in V. (V is “closed under scalar multiplication”.)
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- $(cd)\mathbf{u} = c(d\mathbf{u})$.
- $1\mathbf{u} = \mathbf{u}$.

Column Spaces

Definition. The **column space**, written as Col(A), of an $m \times n$ matrix A is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix}$, then
$\text{Col}(A) = \text{span}(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$.

$$A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} \underline{1} & -10 & -24 & -42 \\ 0 & \underline{2} & 6 & 10 \\ 0 & 0 & \underline{3} & 3 \end{bmatrix}$$

$$\text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix}, \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \right\}$$

How to solve for Col(A):

1. Put matrix A into REF
2. Find all the pivots of A
3. Map the pivots to the columns of your original matrix, A

Null Spaces

Definition. The **nullspace** of an $m \times n$ matrix A, written as $\text{Nul}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$; that is, $\text{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^n \ : \ A\mathbf{v} = \mathbf{0}\}$.

How to solve for Nul(A):

1. Set matrix A into **Augmented Matrix** with zeros on the right (**Ax = 0**)
2. Get A into **RREF**
3. Solve for **x**

$\text{Nul}(A) = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \ldots)$

Linear Independence

Definition. Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are said to be **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$).
We say the vectors are **linearly dependent** if they are not linearly independent.

Theorem 30. *Let A be an $m \times n$ matrix. The following are equivalent:*

- *The columns of A are linearly independent.*
- *$A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.*
- *A has n pivots.*
- *there are no free variables for $A\mathbf{x} = \mathbf{0}$.*

Basis and Dimension

Definition. Let V be a vector space. A sequence of vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ in V is a **basis** of V if

- $V = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$, and
- $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ are linearly independent.

The number of vectors in a basis of V is the **dimension** of V.

Basis and Dim of four subspaces:

Rank [r] : Number of pivots matrix has

Let A be an $m \times n$ matrix with rank r

- dim Nul(A) = n - r
- dim Col(A) = r
- dim Nul(A^T) = m - r
- dim Col(A^T) = r

Orthogonal Complements

Definition. Let W be a subspace of $\mathbb{R}^n$. The **orthogonal complement** of W is the subspace $W^{\perp}$ of all vectors that are orthogonal to W; that is

$$W^{\perp} := \{\mathbf{v} \in \mathbb{R}^n \ : \ \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}.$$

Some helpful theorems:

- $(W^{\perp})^{\perp} = W$
- $\text{Nul}(A) = \text{Col}(A^T)^{\perp}$
- $\text{Nul}(A)^{\perp} = \text{Col}(A^T)$
- $\text{Nul}(A^T) = \text{Col}(A)^{\perp}$

Theorem 43. *Let V be a subspace of $\mathbb{R}^n$. Then $\dim V + \dim V^{\perp} = n$.*

Coordinates

Standard basis (ε):

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Generally, if $\mathbf{v}_1$, $\mathbf{v}_2$, … $\mathbf{v}_p$ are a basis B of vector space V, the coordinate vector of any vector **w** in V is:

$$\mathbf{w}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \qquad \text{if } \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

This coordinate vector is unique!

Change of Basis Matrix

Definition. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of $\mathbb{R}^n$. The **change of basis matrix** $I_{\mathcal{C},\mathcal{B}}$ is the matrix such that for all $\mathbf{v} \in \mathbb{R}^n$

$$I_{\mathcal{C},\mathcal{B}}\mathbf{v}_{\mathcal{B}} = \mathbf{v}_{\mathcal{C}}$$

Matrix allowing us to go from coordinates **mapped in B** to be **mapped onto C**

Theorem 45. *Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b_n})$ be a basis of $\mathbb{R}^n$. Then*

$$I_{\mathcal{E}_n,\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix}$$

That is, for all $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v} = \begin{bmatrix} \mathbf{b}_1 & \ldots & \mathbf{b}_n \end{bmatrix} \mathbf{v}_{\mathcal{B}}.$$

How do we compute change of basis matrix:

$I_{\mathcal{B},\mathcal{C}}$?

What we know:

- $I_{En,B}$ = Matrix that maps **coordinates in B** onto **Standard**
- $I_{B,En}$ = Matrix that maps **coordinates in Standard** onto **B**

- $I_{En,C}$ = Matrix that maps **coordinates in C** onto **Standard**
- $I_{C,En}$ = Matrix that maps **coordinates in Standard** onto **C**

$$I_{\mathcal{B},\mathcal{E}_n} I_{\mathcal{E}_n,\mathcal{C}}$$

From right to left:
We map coordinates **from C into the standard** coordinate plane, **then**, we map the newly acquired **standard coordinates onto B's coordinate plane**

AKA: $I_{\mathcal{B},\mathcal{C}}$

Orthogonal and Orthonormal Bases

Definition. An **orthogonal basis** (an **orthonormal basis**) is an orthogonal set of vectors (an orthonormal set of vectors) that forms a basis.

Theorem 47. *Let $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be an orthogonal basis of $\mathbb{R}^n$, and let $\mathbf{v} \in \mathbb{R}^n$. Then*

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \ldots + \frac{\mathbf{v} \cdot \mathbf{b}_n}{\mathbf{b}_n \cdot \mathbf{b}_n}\mathbf{b}_n.$$

When $\mathcal{B}$ is orthonormal, then $\mathbf{b}_i \cdot \mathbf{b}_i = 1$ for $i = 1, \ldots, n$.

Theorem 48. *Let $\mathcal{U} = (\mathbf{u}_1, \ldots, \mathbf{u_n})$ be an orthonormal basis of $\mathbb{R}^n$. Then*

$$I_{\mathcal{U},\mathcal{E}_n} = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_n \end{bmatrix}^T.$$

Why? An $n \times n$-matrix Q is **orthogonal** if $Q^{-1} = Q^T$

Linear Transformation

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in V$ and all $a, b \in \mathbb{R}$.

$$T(\mathbf{0}_V) = T(0 \cdot \mathbf{0}_V) = 0 \cdot T(\mathbf{0}_V) = \mathbf{0}_W \rightsquigarrow T(\mathbf{0}_V) = \mathbf{0}_W$$

To check linearity for a transformation, we can test with 0, since when we multiply anything by 0, we get 0 back in both spaces

Theorem 50. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $m \times n$ matrix A such that*

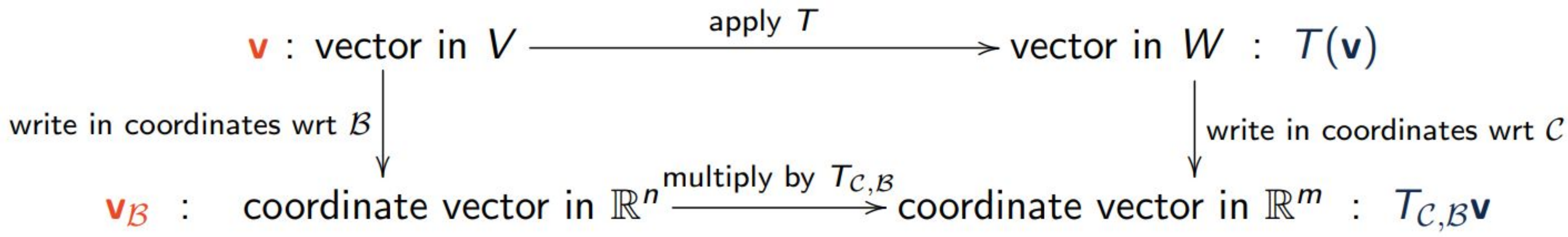
- $T(\mathbf{v}) = A\mathbf{v}$, *for all* $\mathbf{v} \in \mathbb{R}^n$.
- $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$, *where* $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ *is the standard basis of* $\mathbb{R}^n$.

Remark. We call this A the **coordinate matrix of** T with respect to the standard bases - we write $T_{\mathcal{E}_m, \mathcal{E}_n}$.

Coordinate matrices

Theorem 51. *Let V, W be two vector space, let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of V and $\mathcal{C} = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ be a basis of W, and let $T\colon V \to W$ be a linear transformation. Then there is a $m \times n$ matrix $T_{\mathcal{C},\mathcal{B}}$ such that*

- $T(\mathbf{v})_{\mathcal{C}} = T_{\mathcal{C},\mathcal{B}} \mathbf{v}_{\mathcal{B}}$, *for all* $\mathbf{v} \in V$.
- $T_{\mathcal{C},\mathcal{B}} = \begin{bmatrix} T(\mathbf{b}_1)_{\mathcal{C}} & T(\mathbf{b}_2)_{\mathcal{C}} & \ldots & T(\mathbf{b}_n)_{\mathcal{C}} \end{bmatrix}$.



You Try: Linear Transformations

A certain basis of $M_{2\times 2}$ is $\mathcal{M}$ $\quad \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\} = \left\{ \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \right\}$

The transformation Ψ acts on the basis vectors as follows:

$$\Psi(\mathcal{M}_1) = \begin{bmatrix} 9 & 11 \\ 8 & 14 \end{bmatrix}, \Psi(\mathcal{M}_2) = \begin{bmatrix} 42 & 36 \\ 4 & 23 \end{bmatrix}, \Psi(\mathcal{M}_3) = \begin{bmatrix} 5 & 4 \\ 1 & 6 \end{bmatrix}, \Psi(\mathcal{M}_4) = \begin{bmatrix} 30 & 23 \\ 7 & 22 \end{bmatrix}$$

Compute $\Psi\left(\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix}\right)$

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

Partial Solution: Linear Transformations

$$\Psi(\mathcal{M}_1) = \begin{bmatrix} 9 & 11 \\ 8 & 14 \end{bmatrix}, \Psi(\mathcal{M}_2) = \begin{bmatrix} 42 & 36 \\ 4 & 23 \end{bmatrix} \Psi(\mathcal{M}_3) = \begin{bmatrix} 5 & 4 \\ 1 & 6 \end{bmatrix} \Psi(\mathcal{M}_4) = \begin{bmatrix} 30 & 23 \\ 7 & 22 \end{bmatrix}$$

$$\text{Compute } \Psi\left(\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix}\right)$$

Definition. Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(a\mathbf{v} + b\mathbf{w}) = aT(\mathbf{v}) + bT(\mathbf{w})$$

Express $\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix}$ in terms of $\mathcal{M}$ basis vectors: $\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix} = 1\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$\text{Thus: } \Psi\left(\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix}\right) = \Psi(1\mathcal{M}_1 + 1\mathcal{M}_2 + 3\mathcal{M}_3)$$

Solution: Linear Transformations

Express $\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix}$ in terms of $\mathcal{M}$ basis vectors: $\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix} = 1\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 3 & 0 \\ 6 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Thus: $\Psi\left(\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix}\right) = \Psi(1\mathcal{M}_1 + 1\mathcal{M}_2 + 3\mathcal{M}_3)$

Apply Linearity: $\Psi(1\mathcal{M}_1 + 1\mathcal{M}_2 + 3\mathcal{M}_3) = 1\Psi(\mathcal{M}_1) + 1\Psi(\mathcal{M}_2) + 3\Psi(\mathcal{M}_3)$

Substitute Values: $1\Psi(\mathcal{M}_1) + 1\Psi(\mathcal{M}_2) + 3\Psi(\mathcal{M}_3) = \begin{bmatrix} 9 & 11 \\ 8 & 14 \end{bmatrix} + \begin{bmatrix} 42 & 36 \\ 4 & 23 \end{bmatrix} + \begin{bmatrix} 15 & 12 \\ 3 & 18 \end{bmatrix}$

Solve: $\Psi\left(\begin{bmatrix} 7 & 4 \\ 6 & 3 \end{bmatrix}\right) = \begin{bmatrix} 66 & 59 \\ 15 & 55 \end{bmatrix}$

Determinants (how to find them)

2x2: easy formula!

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Triangular: multiply all of the diagonal entries together

Otherwise: cofactor expansion

Note: if the matrix A is not invertible, det(A) = 0 ← this is the definition of a determinant!

Properties of determinants

(Replacement) Adding a multiple of one row to another row *does not change* the determinant.
(Interchange) Interchanging two different rows *reverses the sign* of the determinant.
(Scaling) Multiplying all entries in a row by s, *multiplies* the determinant by s.

These three things also apply to the columns of a matrix!

Let A, B be two $n \times n$-matrices. Then $\det(AB) = \det(A)\det(B)$

If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

Let A be an $n \times n$-matrix. Then $\det(A^T) = \det(A)$

Eigenvectors and Eigenvalues

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

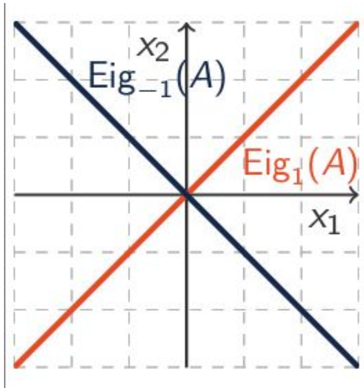
An **eigenvector** of A is a nonzero $\mathbf{v} \in \mathbb{R}^n$

such that

$$A\mathbf{v} = \lambda \mathbf{v}$$

eigenvalue

An eigenspace is all the eigenvectors associated with a specific eigenvalue.



$$A \begin{bmatrix} x \\ x \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ x \end{bmatrix}$$

$$A \begin{bmatrix} -x \\ x \end{bmatrix} = -1 \cdot \begin{bmatrix} -x \\ x \end{bmatrix}$$

Eigenvectors are always linearly independent!

Properties of Eigenvalues and Eigenvectors

For a 2x2 matrix:

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$$

Multiplicity:

- **Algebraic** multiplicity is the multiplicity of λ in the characteristic polynomial
- **Geometric** multiplicity is the dimension of the eigenspace of λ

Trace: the sum of the diagonal entries of a matrix

- Tr(A) = sum of all eigenvalues
- det(A) = product of all eigenvalues

Markov Matrices

$$\begin{bmatrix} 0 & .25 & .4 \\ 1 & .25 & .2 \\ 0 & .5 & .4 \end{bmatrix}$$

Definition: a square matrix with non-negative entries where the sum of terms in each column is 1

A **probability vector** has entries that add up to 1

The λ of a Markov Matrix:

- 1 is always an eigenvalue, and the corresponding eigenvector is called **stationary**
- All other $|\lambda| \le 1$

Diagonalization

$$P = \begin{bmatrix} \mathbf{v_1} & \dots & \mathbf{v_n} \end{bmatrix}$$

v are eigenvectors

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

For a matrix A to be diagonalizable:

- A must be square
- A must have as many unique eigenvectors as rows/columns (i.e. it has an eigenbasis)
- $A = PDP^{-1}$

Observe that

$$A = PDP^{-1} = I_{\mathcal{E}_n,\mathcal{B}} D I_{\mathcal{B},\mathcal{E}_n}$$

Where B is the eigenbasis → diagonalizing is a base change to the eigenbasis

Matrix Powers and Matrix Exponential

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Matrix power: diagonal matrices are easy!

$$A^m = PD^mP^{-1}$$

Where D^m =

$$\begin{bmatrix} (\lambda_1)^m & & \\ & \ddots & \\ & & (\lambda_n)^m \end{bmatrix}$$

Matrix exponential:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots$$

$$e^{At} = Pe^{Dt}P^{-1}$$

Linear Differential Equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

With initial condition:

$$\mathbf{u}(0) = \mathbf{v}$$

Let A be an $n \times n$ *matrix and* $\mathbf{v} \in \mathbb{R}^n$ *The solution of the differential equation* $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ *with initial condition* $\mathbf{u}(0) = \mathbf{v}$ *is* $\mathbf{u}(t) = e^{At}\mathbf{v}$

If $v_1, v_2, \ldots v_n$ is an eigenbasis of A:

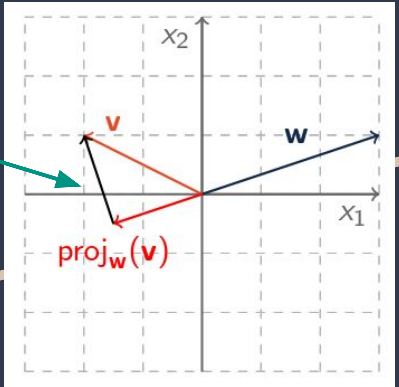
$$e^{At}\mathbf{v} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$$

Vector Projections

Projection of **v** onto **w**

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) := \frac{\mathbf{w}\cdot\mathbf{v}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$$

Error term



Projecting **v** onto **w** yields the vector in span(**w**) that is closest to **v**.

The **error term** is **v** - $\text{proj}_{\mathbf{w}}(\mathbf{v})$ and is in span(**w**)$^{\perp}$

Can also use:

$$\text{proj}_{\mathbf{w}}(\mathbf{v}) = \boxed{\left(\frac{1}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}\mathbf{w}^T\right)}\mathbf{v}$$

Where the boxed term is called the orthogonal projection matrix onto span(**w**)

Subspace Projections

Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$. Then $\mathbf{v}$ can be written uniquely as

$$\mathbf{v} = \underbrace{\hat{\mathbf{v}}}_{\text{in } W} + \underbrace{\mathbf{v}^{\perp}}_{\text{in } W^{\perp}}$$

$\hat{\mathbf{v}}$ is calculated by projecting **v** onto an orthogonal basis of W

P_W is the orthogonal projection matrix for subspace W. Calculate P_W by projecting each column of the identity matrix onto W and join them all in a matrix

$Q = I - P_W$, where I is the identity. Then $P_{W^{\perp}} = Q$

Least Squares Solutions:

Trying to minimize the distance between Ax and b for an inconsistent system

$$A\hat{\mathbf{x}} = \text{proj}_{\text{Col}(A)}(\mathbf{b})$$

LSQ solution

General algorithm:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$$

Find A^T and A^TA, then solve the above system with any method you prefer.

For linear regressions:

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

A: design matrix X

$\hat{\mathbf{x}}$

b: observation vector $\mathbf{y}$

The shape of the design matrix depends on the problem!

Gram-Schmidt Method

Algorithm. (Gram-Schmidt orthonormalization) Given a basis $\mathbf{a}_1, \ldots, \mathbf{a}_m$, produce an orthogonal basis $\mathbf{b}_1, \ldots, \mathbf{b}_m$ and an orthonormal basis $\mathbf{q}_1, \ldots, \mathbf{q}_m$.

$$\mathbf{b}_1 = \mathbf{a}_1, \qquad \mathbf{q}_1 = \frac{\mathbf{b}_1}{\|\mathbf{b}_1\|}$$

$$\mathbf{b}_2 = \mathbf{a}_2 - \underbrace{\mathrm{proj}_{\mathrm{span}(\mathbf{q}_1)}(\mathbf{a}_2)}_{=(\mathbf{a}_2 \cdot \mathbf{q}_1)\mathbf{q}_1}, \qquad \mathbf{q}_2 = \frac{\mathbf{b}_2}{\|\mathbf{b}_2\|}$$

$$\mathbf{b}_3 = \mathbf{a}_3 - \underbrace{\mathrm{proj}_{\mathrm{span}(\mathbf{q}_1, \mathbf{q}_2)}(\mathbf{a}_3)}_{(\mathbf{a}_3 \cdot \mathbf{q}_1)\mathbf{q}_1 + (\mathbf{a}_3 \cdot \mathbf{q}_2)\mathbf{q}_2} \qquad \mathbf{q}_3 = \frac{\mathbf{b}_3}{\|\mathbf{b}_3\|}$$

- QR Decomposition: Let A be an m × n matrix of rank n.
  - There is an m × n-matrix Q with orthonormal columns
  - An upper triangular n × n invertible matrix R such that A = QR.

$$\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{q}_1 & \mathbf{a}_2 \cdot \mathbf{q}_1 & \mathbf{a}_3 \cdot \mathbf{q}_1 \\ 0 & \mathbf{a}_2 \cdot \mathbf{q}_2 & \mathbf{a}_3 \cdot \mathbf{q}_2 \\ 0 & 0 & \mathbf{a}_3 \cdot \mathbf{q}_3 \end{bmatrix}$$

Q R

Spectral Theorem

Theorem 84. *Let A be a symmetric $n \times n$ matrix. Then A has an orthonormal basis of eigenvectors.*

Orthonormal basis: all vectors are orthogonal (perpendicular) to each other and dot product of themselves = 1

Theorem 85. *Let A be a symmetric $n \times n$ matrix. Then there is a diagonal matrix D and a matrix Q with orthonormal columns such that $A = QDQ^T$.*

$$Q^{-1} = Q^T$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$\lambda_1 = 2$: Eigevector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Normalized, we get $\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 4$: Eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Normalized, we get $\mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Singular Value Decomposition

Definition. Let A be an $m \times n$ matrix. A **singular value decomposition** of A is a decomposition $A = U\Sigma V^T$ where

- U is an $m \times m$ matrix with orthonormal columns,
- Σ is an $m \times n$ rectangular diagonal matrix with non-negative numbers on the diagonal,
- V is an $n \times n$ matrix with orthonormal columns.

Remark. The diagonal entries $\sigma_i = \Sigma_{ii}$ which are positive are called the singular values of A. We usually arrange them in decreasing order, that is $\sigma_1 \geq \sigma_2 \geq \dots$

SVD Algorithm

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$

Eigenbasis: $\underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}}_{:=\mathbf{v}_1}, \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}}_{:=\mathbf{v}_2}, \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}_{:=\mathbf{v}_3}$

Singular values: $\sigma_1 := \sqrt{3}, \sigma_2 = 1$

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{u}_1 := \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\frac{3}{\sqrt{6}} \\ \frac{3}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\rightsquigarrow U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Let A be an mxn matrix with rank r

1. Find orthonormal eigenbasis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of A^TA with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = 0 = \cdots = \lambda_n$.
2. Set $\sigma_i = \sqrt{\lambda_i}$ for $i = 1, \ldots, n$.
3. Set $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1, \ldots, \mathbf{u}_r = \frac{1}{\sigma_r} A\mathbf{v}_r$
4. Find $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m \in \mathbb{R}^m$ such that $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is an orthonormal basis of $\mathbb{R}^m$
5. $U = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_m \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{\min\{m,n\}} \end{bmatrix}$, $V = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$

Theorem 86. *Let A be an $m \times n$ matrix with rank r, and let $U = \begin{bmatrix} \mathbf{u}_1 & \ldots \mathbf{u}_m \end{bmatrix}$, $V = \begin{bmatrix} \mathbf{v}_1 & \ldots \mathbf{v}_n \end{bmatrix}$, Σ be such that $A = U\Sigma V^T$ is a SVD of A. Then*

- $(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ *is a basis of* $\mathrm{Col}(A)$.
- $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ *is a basis of* $\mathrm{Col}(A^T)$.
- $(\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m)$ *is a basis of* $\mathrm{Nul}(A^T)$.
- $(\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n)$ *is a basis of* $\mathrm{Nul}(A)$.

You Try: SVD

$$A = U\Sigma V^T$$

where: $A = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$

$$V = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

Solve for the matrices U and Σ

Hint: $A^T A$ has $\lambda_1 = 9, \lambda_2 = 1$

SVD Algorithm:

Let A be an mxn matrix with rank r

1. Find orthonormal eigenbasis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of $A^T A$ with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = 0 = \cdots = \lambda_n$.
2. Set $\sigma_i = \sqrt{\lambda_i}$ for $i = 1, \ldots, n$.
3. Set $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1$, …, $\mathbf{u}_r = \frac{1}{\sigma_r} A\mathbf{v}_r$
4. Find $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m \in \mathbb{R}^m$ such that $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$
5.

$$U = \begin{bmatrix} \mathbf{u}_1 & \ldots & \mathbf{u}_m \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{\min\{m,n\}} \end{bmatrix}, V = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$$

Solution: SVD

$I)\ A^TA = \begin{bmatrix}0 & 2 & 1\\1 & 2 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 2\\1 & 0\end{bmatrix} = \begin{bmatrix}5 & 4\\4 & 5\end{bmatrix}$

$II)$ Find eigenvalues: $p_A(\lambda) = (5-\lambda)^2 - 16 = (\lambda-9)(\lambda-1)$

$\lambda = 9, 1 \Rightarrow \sigma = 3, 1$

$III)$ With known singular values: $\Sigma = \begin{bmatrix}\sigma_1 & 0\\0 & \sigma_2\\0 & 0\end{bmatrix} = \begin{bmatrix}3 & 0\\0 & 1\\0 & 0\end{bmatrix}$

$IV)\ \vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{3}\frac{\sqrt{2}}{2}\begin{bmatrix}0 & 1\\2 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix} = \frac{\sqrt{2}}{6}\begin{bmatrix}-1\\-4\\-1\end{bmatrix}$

$V)\ \vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_1 = \frac{1}{1}\frac{\sqrt{2}}{2}\begin{bmatrix}0 & 1\\2 & 2\\1 & 0\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$VI)$ calculating a 3rd orthonormal vector: $\vec{u}_3 = \frac{1}{3}\begin{bmatrix}2\\-1\\2\end{bmatrix}$

$VII)\ U = \begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \vec{u}_3\end{bmatrix}$

Low rank approximation via SVD

Theorem 87. *Let A be an $m \times n$ matrix with rank r, and let $U = \begin{bmatrix} \mathbf{u}_1 & \dots \mathbf{u}_m \end{bmatrix}$, $V = \begin{bmatrix} \mathbf{v}_1 & \dots \mathbf{v}_n \end{bmatrix}$ be matrices with with orthonormal columns and Σ be a rectangular diagonal $m \times n$ matrix such that $A = U\Sigma V^T$ is an SVD of A. Then*

$$\begin{aligned} A &= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T \\ &= \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_r \end{bmatrix}^T \end{aligned}$$

Compact SVD

For $k \leq r$, define

$$A_k = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T$$

If $\sigma_1 \gg \sigma_2 \gg \ldots\ldots$, Then, A_k is a good approx. since most of the information is inside of first term

Definition. Let A be an $m \times n$ matrix with rank r. A **compact singular value decomposition** of A is a decomposition $A = U_c \Sigma_c V_c^T$ where

- $U_c = \begin{bmatrix} \mathbf{u}_1 & \ldots \mathbf{u}_r \end{bmatrix}$ is an $m \times r$ matrix with orthonormal columns,
- Σ_c is an $r \times r$ diagonal matrix with positive diagonal elements,
- $V_c = \begin{bmatrix} \mathbf{v}_1 & \ldots \mathbf{v}_r \end{bmatrix}$ is an $n \times r$ matrix with orthonormal columns.

rank = 2

$$\text{SVD} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Pseudo Inverse

Definition. Let A be an $m \times n$ matrix with rank r. Given the compact singular value decomposition $A = U_c \Sigma_c V_c^T$ where

- $U_c = \begin{bmatrix} \mathbf{u}_1 & \dots \mathbf{u}_r \end{bmatrix}$ is an $m \times r$ matrix with orthonormal columns,
- Σ_c is an $r \times r$ diagonal matrix with positive diagonal elements,
- $V_c = \begin{bmatrix} \mathbf{v}_1 & \dots \mathbf{v}_r \end{bmatrix}$ is an $n \times r$ matrix with orthonormal columns,

pseudoinverse A^+ of A as $V_c \Sigma_c^{-1} U_c^T$.

Theorem 88. *Let* $\mathbf{v} \in \mathrm{Col}(A^T)$ *and* $\mathbf{w} \in \mathrm{Col}(A)$*. Then* $A^+ A\mathbf{v} = \mathbf{v}$ *and* $AA^+\mathbf{w} = \mathbf{w}$

Theorem 89. *Let* A *be an* $m \times n$ *matrix and let* $\mathbf{b} \in \mathbb{R}^m$*. Then* $A^+\mathbf{b}$ *is the LSQ solution of* $A\mathbf{x} = \mathbf{b}$ *(with minimum length).*

Tips on Approaching Conceptual Questions

What topic is the question asking about?

Not always immediately apparent. Think about the relevant theorems: is it a vector or a matrix? Is the matrix invertible? Is the matrix orthonormal?

For true/false questions:

Look for counter examples. Try easy test cases like $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ to prove false

If you think the statement is true, try to connect theorems by breaking down what each part of the statement means

Conceptual Question Example 1

17. (5 points) Let A be an $n \times n$ matrix. Consider the following statements:

(T1) If A is not the zero matrix, then A^2 is also not the zero matrix.

(T2) If A is invertible, then A^2 is also invertible.

Which of the statements are ALWAYS TRUE?

(A) Neither Statement (T1) nor Statement (T2).

(B) Both Statement (T1) and Statement (T2).

(C) Only Statement (T2).

(D) Only Statement (T1).

Topic: matrix multiplication and matrix inverses

T1: FALSE. Look for a counterexample – pick something 2x2 with a lot of zeros, such as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

T2: TRUE. To be invertible, a matrix has to have a non-zero determinant

- Recall properties of determinants: $\det(A^2) = \det(A)^2$
- If $\det(A)$ is non-zero, so is $\det(A^2)$

Conceptual Question Example 2

18. (5 points) Let A be a diagonalizable 3×3 matrix with only two distinct eigenvalues. Which of the following statements is FALSE?

(A) The matrix $5A$ is diagonalizable.

(B) The matrix A has an eigenbasis.

(C) There are no more than two linearly independent eigenvectors of A.

(D) There is an eigenvalue of A for which the corresponding eigenspace is spanned by two linearly independent eigenvectors.

Topic: diagonalization, eigenvectors

A: TRUE. Look at the definition of an eigenvalue: $A\mathbf{v} = \lambda\mathbf{v}$ – multiplying A by 5 does not change the vector itself so 5A will also be diagonalizable

B: TRUE. This is part of the definition of diagonalizable matrices

C: FALSE. If A has an eigenbasis, it must have **three** linearly independent eigenvectors

D: TRUE. If there are only two distinct eigenvalues but there is an eigenbasis, one of the eigenspaces must have a span of 2 eigenvectors

Professor's Tips (From Spring 2024)

- Realize how important REF & RREF are
  - Help to solve linear systems
  - Number of pivots can help you solve for the dimension of all fundamental subspaces
  - If RREF([A]) = [I], [A] is invertible
  - REF serves as a test for linear independence and solves for the basis of the column space
- Apply your understanding of the fundamental subspaces wherever possible
  - Column space is the possible solution set of a linear system
  - Null space is the 0-eigenspace
  - Rank-nullity is important for conceptual questions
  - All decomposition, span, and transformation questions are fundamental subspace questions
  - Know orthogonal complements

Professor’s Tips (From Spring 2024)

- Abstract vector space questions may seem intimidating, but each vector can be written as a column vector and handled normally
  - Matrices, polynomials, etc.
- There are an infinite number of bases for a subspace. How we express a vector is up to us
  - There is nothing special about standard basis except for being convenient
- There is an underlying geometry to all operations in linear algebra
  - Can be used to understand coordinate transforms, determinants, bases, subspaces, decompositions, and more
- Conceptual questions can be solved with applying basic formulas & theorems and inspecting the consequences of them

Questions?

Good luck on your final exam!

Join the queue to see the past worksheets and solutions!