MATH 285 Midterm 3 Review

CARE

Disclaimer

- These slides were prepared by tutors that have taken Math 285. We believe that the concepts covered in these slides could be covered in your exam.
- HOWEVER, these slides are NOT comprehensive and may not include all topics covered in your exam. These slides should not be the only material you study.
- While the slides cover general steps and procedures for how to solve certain types of problems, there will be exceptions to these steps. Use the steps as a general guide for how to start a problem but they may not work in all cases.

Topics

I. Systems of Differential Equations
   I. System notation
   II. Variation of Parameters
   III. Eigenvectors/Eigenvalues
   IV. Diagonalization
   V. Putzer's Method
II. Boundary Value Problems
   I. Eigenfunction Problems
III. Fourier Series

Systems of Ordinary Differential Equations

- Many **physical phenomena** can be described by a coordinated system of differential equations
- For example, **Maxwell's Equations**:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{B}$$

- Also, **higher order differential equations** can be broken down into **systems** of ODE's

Creating Systems

- General process:
  - **Redefine a derivative** as a new variable
  - Create **vectors** $\boldsymbol{v}$**,** $\boldsymbol{g}$**,** and the **matrix** $\boldsymbol{A}$
  - Write the differential equation in general form:

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(t)\boldsymbol{v} + \boldsymbol{g}(t)$$

Existence and Uniqueness

- If the equation is **linear**:

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{t})$$

  - A **unique solution exists** if $A(t)$ **and** $g(t)$ **are defined** on the interval

The Fundamental Solution Matrix

- We want to find $\boldsymbol{n}$ **solutions** to the system of differential equations, where **each solution** $\boldsymbol{v_i(t)}$ **is an** $\boldsymbol{n}$**-vector**
- Build the fundamental solution matrix $M(t)$ by **column-stacking each solution** $\boldsymbol{v_i(t)}$

$$\boldsymbol{v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}} \qquad \boldsymbol{v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}} \qquad \mathbf{M} = \begin{bmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} \\ \boldsymbol{y_1} & \boldsymbol{y_2} \end{bmatrix}$$

The Fundamental Solution Matrix

- The **Wronskian** is the **determinant of** $\boldsymbol{M(t)}$
  - If $\boldsymbol{W(t)} \neq \boldsymbol{0}$, then a solution exists
  - If coefficients are continuous, then $\boldsymbol{W(t)}$ is either identically zero or nonzero

- $M(t)$ depends on how the solutions are chosen
  - **Most convenient choice is $\boldsymbol{M(t_0) = I}$**
  - $\boldsymbol{M(t_0) = I}$ **can be calculated as $\boldsymbol{M(t)M^{-1}(t_0)}$**

Putting it Together

- The **homogeneous solution** to the following differential equation:

$$\frac{dv}{dt} = A(t)v \qquad v(t_0) = v_0$$

is given by:

$$v(t) = M(t)M^{-1}(t_0)v_0$$

Abel's Theorem Extended

- For a system of linear differential equations given by

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{v}$$

$$\frac{d\boldsymbol{W}}{dt} = \mathbf{Tr}\big(\boldsymbol{A}(\boldsymbol{t})\big)\boldsymbol{W}$$

  - ***W*** is the **Wronskian** / determinant of the fundamental solution matrix
  - The **"trace" (Tr)** of a matrix is the **sum of its diagonal components**
- If $A(t)$ is continuous, then the **Wronskian is either always or never 0**

Variation of Parameters

- For a system of differential equations given by:

$$\frac{dv}{dt} = A(t)v + g(t)$$

- The **general solution** is given by:

$$v(t) = \underbrace{M(t)\int_{t_0}^{t} M^{-1}(s)g(s)ds}_{\textbf{particular solution}} + \underbrace{M(t)M^{-1}(t_0)v_0}_{\textbf{characteristic solution}}$$

Calculating Eigenvalues and Eigenvectors

- **Eigenvalues (λ)** and **eigenvectors (v)** are given by:

$$(\boldsymbol{A} - \boldsymbol{\lambda I})\boldsymbol{v} = \mathbf{0}$$

- Calculate **eigenvalues** with: $\boldsymbol{det}(\boldsymbol{A} - \boldsymbol{\lambda I}) = \mathbf{0}$

- Then, calculate **eigenvectors** with **first equation**

Matrix Exponentials

- The matrix exponential definition comes from the **power series** definition of an exponent:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{A^2 t^2}{2!} + \cdots$$

Constant Coefficient Linear Systems

- If the matrix A is **no longer a function of t**:

$$\frac{d\boldsymbol{v}}{d\boldsymbol{t}} = \boldsymbol{A}\boldsymbol{v}$$

- The **fundamental solution matrix** can be calculated by the **matrix exponential**:

$$\boldsymbol{M}(\boldsymbol{t}) = \boldsymbol{e}^{\boldsymbol{A}\boldsymbol{t}}$$

Diagonalization

- If there are n **linearly independent eigenvectors**, then the matrix exponential can be calculated as:

$$e^{At} = UDU^{-1}$$

eigenvalue

$$U = [v_1 \quad v_2 \quad \ldots], \qquad D = \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & \ldots \end{bmatrix}$$

eigenvector

Putzer's Method

- Putzer's Method will ***always* work** for solving the matrix exponential
- Trying to get to:

$$e^{At} = \boldsymbol{B_0 r_1 + B_1 r_2 + B_2 r_3 \ldots}$$

- **Number of terms matches degree of matrix**, I.E. 2x2 matrix goes up to the $\boldsymbol{B_1 r_2}$ term

Putzer's Method Contd.

- First, **calculate the B matrices**
- Follow the pattern:

$$\boldsymbol{B_0 = I}$$

$$\boldsymbol{B_1 = (A - \lambda_1 I)B_0}$$

$$\boldsymbol{B_2 = (A - \lambda_2 I)B_1}$$

$$\dots$$

Putzer's Method Contd.

- Second, **calculate the r functions**, then plug everything in
- Follow the pattern:

$$\frac{dr_1}{dt} = \lambda_1 r_1, \qquad r_1(0) = 1$$

$$\frac{dr_2}{dt} = \lambda_2 r_2 + r_1, \qquad r_2(0) = 0$$

$$\frac{dr_3}{dt} = \lambda_3 r_3 + r_2, \qquad r_3(0) = 0$$

$$\cdots$$

Putzer’s Method Notes and Example

- Putzer’s Method **doesn’t require knowing the eigenvectors**
- More calculation heavy than diagonalization, but **no matrix multiplication** either
- Example:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Putzer's Method Notes and Example

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B_1 = A - 0I = A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$B_2 = (A - 2I)\, B_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Putzer's Method Notes and Example

$$\frac{dr_1}{dt} = 0r_1 \qquad\qquad r_1(0) = 1$$

$$r_1(t) = e^{0t} = 1$$

$$\frac{dr_2}{dt} = 2r_2(t) + 1 \qquad\qquad r_2(0) = 0$$

$$r_2(t) = \frac{e^{2t}}{2} - \frac{1}{2}$$

Ordinarily we would have to compute the solution to

$$\frac{dr_3}{dt} = 2r_3 + r_2(t) = 2r_3 + \frac{e^{2t}}{2} - \frac{1}{2} \qquad r_3(0) = 0$$

Putzer's Method Notes and Example

$$e^{tA} = 1 \cdot \boldsymbol{B}_0 + \frac{e^{2t} - 1}{2} \boldsymbol{B}_1 = \begin{pmatrix} \frac{e^{2t}+1}{2} & \frac{e^{2t}-1}{2} & 0 \\ \frac{e^{2t}-1}{2} & \frac{e^{2t}+1}{2} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}$$

Boundary Value Problems

- A boundary value problem is an **analog to an initial value problem**
- Instead of specifying just an initial condition, **multiple “boundary” values are specified** to constrain the solution

Eigenvalue Problems

- Eigenvalue problems are boundary value problems with an **unknown parameter λ**
- We look for cases that have **non-trivial (non-zero) solutions**
  - **Generally a good idea to look at the positive case first**
- Final answer is a set of **eigenvalues** and the **eigenfunctions** (solution functions) that go with it

Eigenvalue Problem Example

$$y'' + \lambda y = 0 \qquad \lambda = k^2 \qquad y(0) = 0 \qquad y'(1) = 0$$

Case 1:	$\lambda > 0$	$y = A\cos(kx) + B\sin(kx)$	$\lambda = \left(n + \frac{1}{2}\right)^2 \pi^2$
Case 2:	$\lambda = 0$	$y = A + Bx$	$\lambda = 0$
Case 3:	$\lambda < 0$	$y = Ae^{kx} + Be^{-kx}$	$\lambda = 0$

5.3 Eigenvalue Problem Example

- We first check **λ > 0**, which has general solution:

$$y(x) = A\cos(kx) + B\sin(kx)$$

- Applying our first Boundary Condition

$$y(0) = A\cos 0 + B\sin 0 = A = 0$$

5.3 Eigenvalue Problem Example

- Since A = 0,

$$y(x) = B\sin(kx)$$

- Applying our next boundary condition,

$$y'(1) = Bk\cos(k) = 0$$

- Since we want a non-trivial answer, we do not want B or k to be 0. Therefore, we solve cos(k) = 0, which has solutions as follows:

$$k = \frac{(2n+1)\pi}{2}, \qquad n = 0, 1, 2, \ldots$$

5.3 Eigenvalue Problem Example

- By our earlier definitions:

$$\lambda_n = k_n^2 = \left(\frac{(2n+1)\pi}{2}\right)^2$$

and…

$$y_n(x) = \sin(k_n x) = \sin\left(\frac{(2n+1)\pi}{2}x\right)$$

5.3 Eigenvalue Problem Example

- We next check **λ = 0**, which has general solution:

$$y(x) = Ax + B$$

- Applying the first boundary condition:

$$y(0) = 0 \Rightarrow B = 0, \text{ so } y(x) = Ax$$

- Applying the second boundary condition:

$$y'(x) = A. \text{ Apply } y'(1) = 0 \Rightarrow A = 0.$$

- So the only solution is y = 0

5.3 Eigenvalue Problem Example

- We next check **λ < 0**, which has general solution:

$$y(x) = Ae^{kx} + Be^{-kx}$$

- Using the first boundary condition,

$$y(0) = Ae^0 + Be^0 = A + B = 0 \quad \Rightarrow \quad B = -A$$

- Applying the second boundary condition:

$$y'(1) = 2Ak\cosh(k) = 0.$$

- This also implies that A = 0 because k and cos(k) are not zero, which means that A = B = 0

Fourier Series

- (Nearly) any periodic function can be represented as an **infinite series of sin and cos functions**
- A Fourier Series **will always repeat periodically**, even if the modelled function is only defined on a certain interval

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Fourier Coefficients

- The coefficients of the series can be directly calculated using the orthogonality of sin and cos
- The integral **bounds are one period**
  - L is half of a period
- **For odd functions, $A_n = 0$**
- **For even functions, $B_n = 0$**

$$A_0 = \frac{1}{2L}\int_0^{2L} f(x)dx$$

$$A_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Cosine Series

- The Fourier cosine series models the even extension of a function
- f(-x) = f(x)
- Note that for the cosine series **L is the full period** (because it's half the period when extended)

$$A_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

Fourier Sine Series

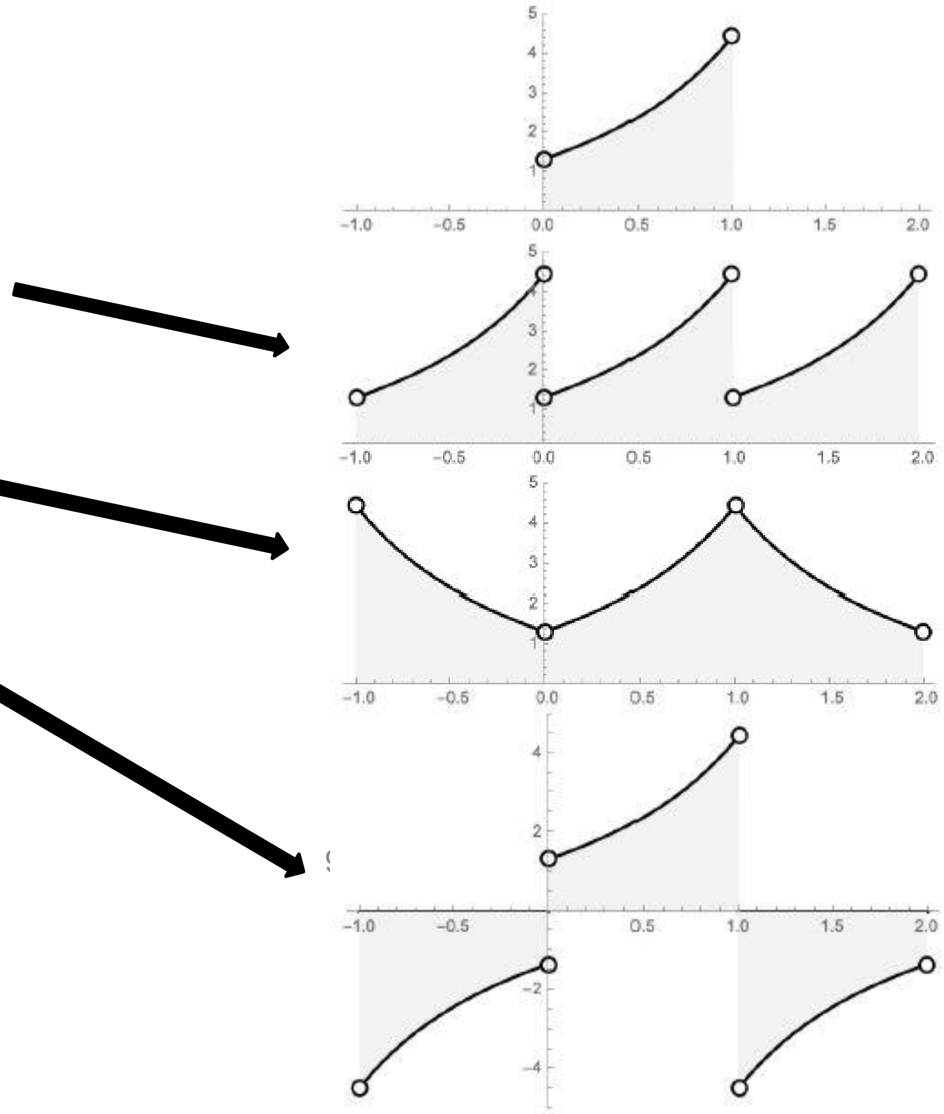
- The Fourier sine series models the odd extension of a function
- f(-x) = -f(x)
- Note that for the sine series **L is the full period** (because it's half the period when extended)

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Fourier Summary

- The **"normal"** Fourier series is the **periodic extension**
- The Fourier **cosine** series is the **even extension**
- The Fourier **sine** series is the **odd extension**



Fourier Convergence Theorem

Theorem 6.2.2. *Suppose that $f(x)$ is piecewise C^2 (twice differentiable) and 2L-periodic: $f(x+2L) = f(x)$, with jump discontinuities at the points of discontinuity. Then the Fourier series*

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

converges to $f(x)$ at points of continuity of $f(x)$, and to $\frac{1}{2}\left(f(x^-) + f(x^+)\right)$ at the jump discontinuities.

- If the modelled function is **continuous**, the series converges to the **function values**
- If the modelled function has a **discontinuity**, the series converges to the **average of the values at the jump**

Thanks for Coming!