Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 221 – Calculus I

Midterm 4 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: November 17, 6:00pm-7:50pm Jiya, Patrick & Emma

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/1056
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

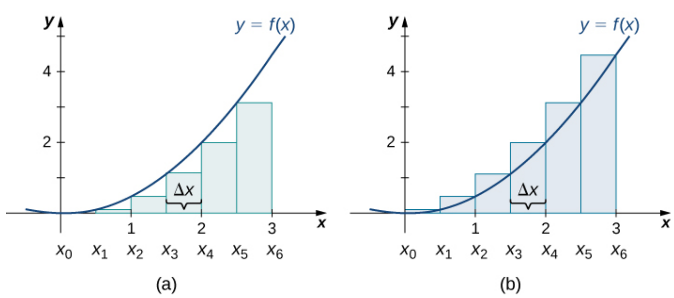
1. Complete the following common properties of sigma notations:

#	Expression	Answers
1	$\sum_{i=1}^{n} c$	
2	$\sum_{i=1}^{n} ca_i$	
3	$\sum_{i=1}^{n} i$	
4	$\sum_{i=1}^{n} i^2$	
5	$\sum_{i=1}^{n} i^3$	

Here are the correct solutions for the expressions:

#	Expression	Answers
1	$\sum_{i=1}^{n} c = nc$	nc
2	$\sum_{i=1}^{n} ca_i$	$c\sum_{i=1}^{n} a_i$
3	$\sum_{i=1}^{n} i$	$\frac{n(n+1)}{2}$
4	$\sum_{i=1}^{n} i^2$	$\frac{n(n+1)(2n+1)}{6}$
5	$\sum_{i=1}^{n} i^3$	$\frac{n^2(n+1)^2}{4}$

2. Use both the left-endpoint and right-endpoint approximations to approximate the area under the curve $f(x) = \frac{x^2}{2}$ in the interval $[0, 3]$.
The two methods of approximating area by (a) left endpoints and (b) right endpoints are in the figure below.



Using 6 sub-intervals, the width of each sub-interval is given by

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{2} = 0.5$$

(a) Left Endpoint

$$Area = \sum_{i=1}^{6} f(x_{i-1})\Delta x$$

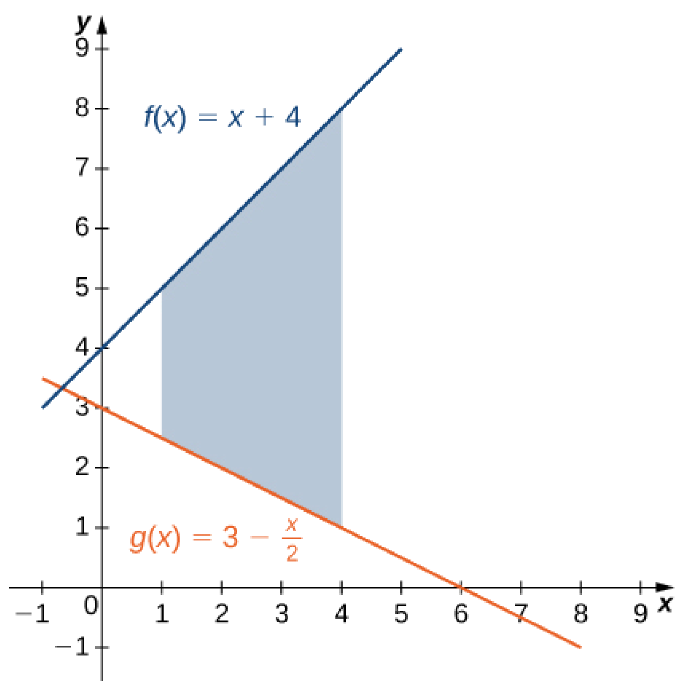
$$= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$

$$= f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5$$
$$= (0)\cdot 0.05 + (0.125)\cdot 0.05 + (0.5)\cdot 0.05 + (1.125)\cdot 0.05 + (2)\cdot 0.05 + (3.125)\cdot 0.05$$
$$= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625$$
$$= 3.4375$$

(a) Right Endpoint

$$Area = \sum_{i=1}^{6} f(x_i)\Delta x$$
$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$
$$= f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 + f(2.5)\cdot 0.5 + f(3)\cdot 0.5$$
$$= (0.125)\cdot 0.05 + (0.5)\cdot 0.05 + (1.125)\cdot 0.05 + (2)\cdot 0.05 + (3.125)\cdot 0.05 + (4.5)\cdot 0.05$$
$$= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25$$
$$= 5.6875$$

3. If the R is the region bounded above by the graph of function $f(x) = x+4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval [1,4], find the area of region R.
The graph of the region R is shown below:



$$Area = \int_a^b [f(x) - g(x)]\,dx$$
$$= \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right] dx$$

$$= \int_1^4 \left[\frac{3x}{2} + 1\right] d$$

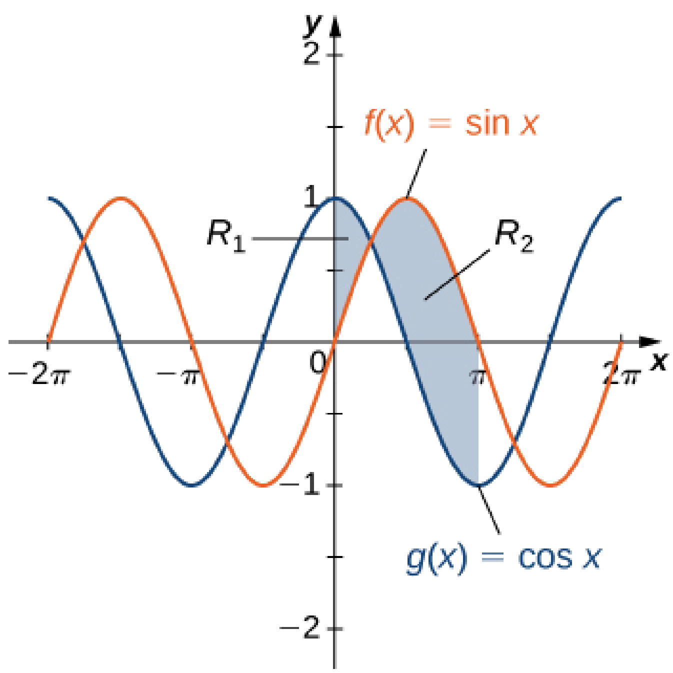
$$= \left[\frac{3x^2}{4} + x\right]\Big|_1^4$$

$$= \left(16 - \frac{7}{4}\right) dx$$

$$\frac{57}{4} units^2$$

4. If the R is the region bounded by the graph of function $f(x) = sin(x)$ and $g(x) = cos(x)$ over the interval $[0, \pi]$, find the area of region R.

The graph of the region R is shown below:



The functions intersect at $x = \pi/4$ For the interval $[0, \pi/4]$, $sin(x) \leq cos(x)$

$$|f(x) - g(x)| = |sin(x) - cos(x)| = cos(x) - sin(x)$$

For the interval $[\pi/4, \pi]$, $sin(x) \geq cos(x)$

$$|f(x) - g(x)| = |sin(x) - cos(x)| = sin(x) - cos(x)$$

$$Area = \int_a^b |f(x) - g(x)| dx$$

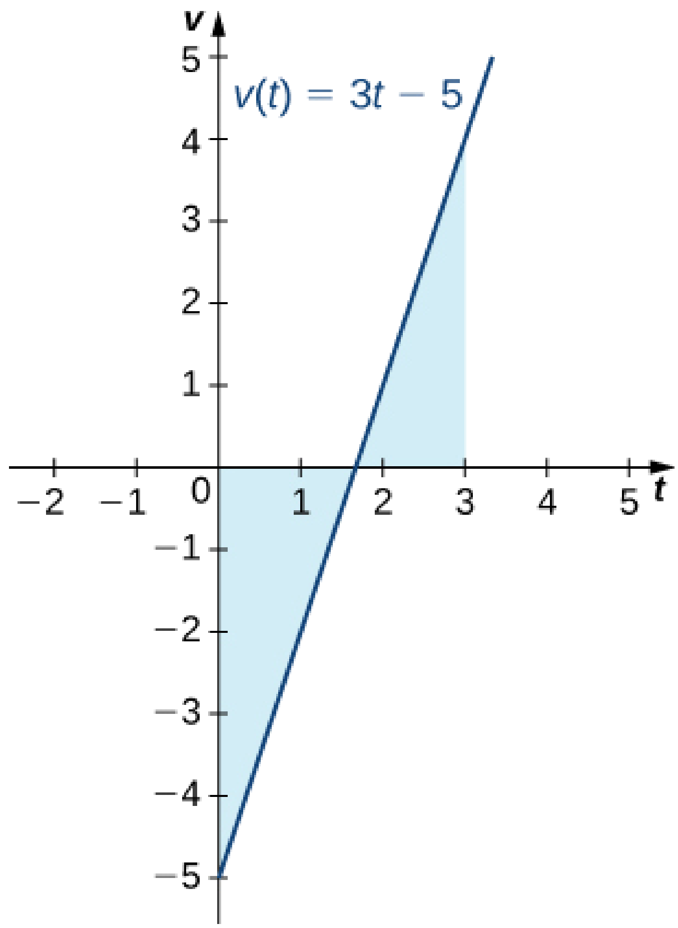
$$= \int_0^\pi |sin(x) - cos(x)|$$

$$= \int_0^{\pi/4} (cos(x) - sin(x))dx + \int_{\pi/4}^{\pi} (sin(x) - cos(x))dx$$

$$= [sin(x) + cos(x)]\Big|_0^{\pi/4} + [-cos(x) - sin(x)]\Big|_{\pi/4}^{\pi}$$

$$= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}$$

5. Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.
The velocity versus time for this function is shown below:



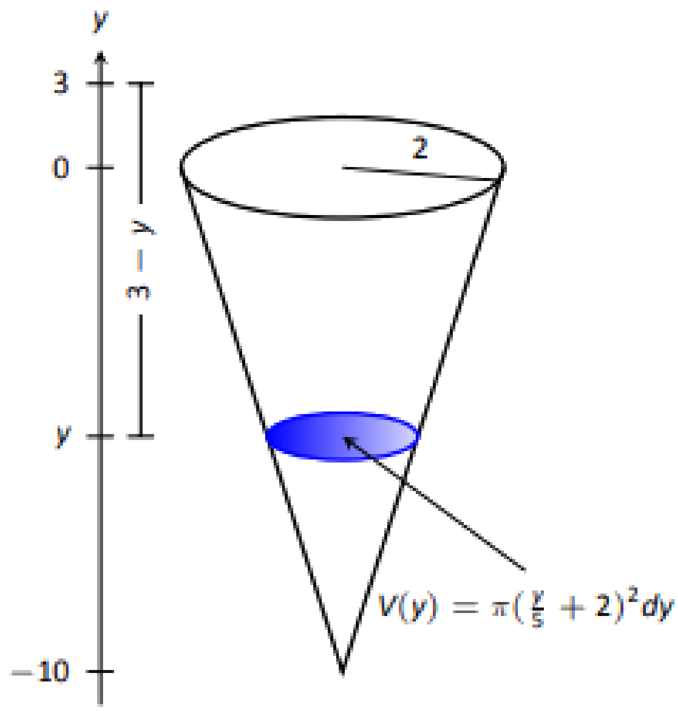
Applying net change theorem:

$$\int_0^3 (3t - 5)dt = \left(\frac{3t^2}{2} - 5t\right)\Bigg|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0$$

$$= \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}$$

6. A conical water tank has its top at ground level and its base 10 feet below ground. The radius of the cone at ground level is 2 ft. It is filled with water weighing $62.4lb/ft^3$ and is to be emptied by pumping the water to a spigot 3 feet above ground level. Find the total amount of work performed in emptying the tank.

The conical tank can be sketched as below. The base of the tank is at $y = 0$ and the top of the tank at $y = 10$.



The figure also sketches a differential element, a cross-sectional circle. The radius of this circle is variable, depending on y. When $y = -10$, the circle has radius 0; when $y = 0$, the circle has radius 2. These two points, $(-10, 0)$ and $(0, 2)$, allow us to find the equation of the line that gives the radius of the cross-sectional circle, which is $r(y) = \frac{1}{5}y + 2$. Hence the volume of water at this height is $V(y) = \pi(\frac{1}{5}y + 2)^2 dy$, where dy represents a very small height of the differential element. The force required to move the water at height y is $F(y) = 62.4 \times V(y)$.

The distance the water at height y travels is given by $h(y) = 3 - y$. Thus the total work done in pumping the water from the tank is

$$W = \int_{-10}^{0} 62.4\pi(\frac{1}{5}y + 1)^2(3 - y)dy$$

$$= 62.4\pi \int_{-10}^{0} (\frac{1}{25}y^3 - \frac{17}{25}y^2 - \frac{8}{5}y + 12)dy$$

$$= 62.4\pi \cdot \frac{220}{3} \approx 14,376 ft - lb$$