Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 231E – Engineering Calculus

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Nov. 15, 2-3:20pm Norita and Yash

Session 2: Nov. 16, 3-4:20pm Grace and Sushrut

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Test the following improper integral for convergence. If it converges, find the value

$$\int_0^\infty e^{-\frac{4}{3}y}dy$$

(a) Diverges
(b) Converges, value is $-\frac{3}{4}$
(c) Converges, value is $\frac{4}{3}$
(d) Converges, value is $\frac{3}{4}$
(e) None of the above

This is an improper integral, so we need to check for convergence.

$$\lim_{t\to\infty}\int_0^t e^{-\frac{4}{3}y}dy$$

$$\lim_{t\to\infty}\left(\frac{-3}{4}e^{\frac{-4}{3}y}\right)\Big|_0^t$$

$$\lim_{t\to\infty}\left(\left(\frac{-3}{4}e^{-\frac{4}{3}t}\right)-\left(-\frac{3}{4}e^{0}\right)\right)$$

$$\lim_{t\to\infty}\frac{-3}{4e^{\frac{4}{3}t}}+\frac{3}{4}$$

$$\frac{-3}{\infty}+\frac{3}{4}=\boxed{\frac{3}{4}}$$

2. Which of the following integrals best represents the length of the curve $y=\cos(x)$ between $x=0$ and $x=1$?

(a) $\int_0^1\sqrt{1+\cos^2(x)}dx$
(b) $\int_0^1\sqrt{1+\sin^2(x)}dx$
(c) $\int_0^1 x\sqrt{1+\cos^2(x)}dx$
(d) $\int_0^1 x\sin(x)$
(e) $\int_0^1 x\sqrt{1+\sin^2(x)}dx$

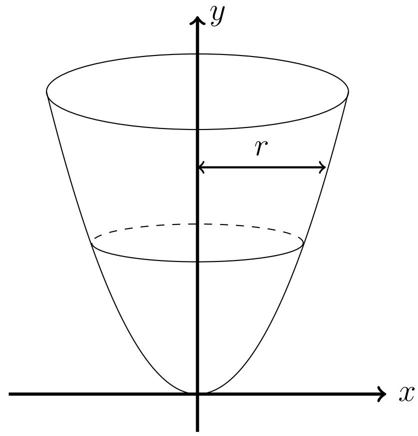
$$ds=\sqrt{dx^2+dy^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$S=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$S = \int_0^1 \sqrt{1 + (-\sin(x))^2}dx$$

$$\boxed{S = \int_0^1 \sqrt{1 + \sin^2(x)}dx}$$

3. The figure below represents the calculation of the volume of solid of revolution. Which of the following integrals most closely represents the calculation of the volume as represented in the figure?



(a) $\int_0^1 \pi y^2 dx$
(b) $\int_0^1 \pi x^2 dy$
(c) $\int_0^1 2\pi xy dx$
(d) $\int_0^1 2\pi xy dy$
(e) None of the above

Use the following formula to find volume: $V = \int A(\frac{dy}{dx})$

From the figure, it can be seen that each component of the volume would be a disk of radius x, and they would stack up along the y-direction, therefore: Area $= \pi r^2 = \pi x^2$, with thickness $= dy$.

$$V = \int \pi x^2 dy$$

The bounds of the function are: $y = 0$ to $y = 1$

$$\boxed{V = \int_0^1 \pi x^2 dy}$$

4. Compute the arc length of the function $y = 1 + 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$

(a) $\frac{14}{9}$
(b) $\frac{10}{9}$
(c) $\frac{2}{9}\sqrt{10}$
(d) $\frac{2}{27}(10\sqrt{10} + 1)$
(e) None of the above

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$y = 1 + 2x^{\frac{3}{2}}, \frac{dy}{dx} = 3x^{\frac{1}{2}}$$

$$S = \int_0^1 \sqrt{1 + (3x^{\frac{1}{2}})^2} dx$$

$$S = \int_0^1 \sqrt{1 + 9x} dx$$

$u = 1 + 9x$, $du = 9dx$, $dx = \frac{1}{9}du$
The u-bounds are: $u = 1 + 9(1) = 10$, $u = 1 + 9(0) = 1$

$$S = \int_0^1 \sqrt{u}\frac{1}{9} du$$

$$S = \frac{1}{9}\left(\frac{2}{3}\right) u^{\frac{3}{2}}\Big|_1^{10}$$

$$S = \left(\frac{2}{27}\right)\left(10^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$S = \left(\frac{2}{27}\right)(10\sqrt{10} - 1)$$

Therefore, the answer is **(e)**.

5. For which value of C does the following series converge?

$$\sum_{n=3}^{\infty}\left(\frac{C}{n+2}-\frac{n}{3n^2+1}\right)$$

(a) It does not converge for any value of C.
(b) $\frac{1}{2}$
(c) 1
(d) $\frac{1}{3}$
(e) 0

$$\frac{C}{n+2}-\frac{n}{3n^2+1}=\frac{C(3n^2+1)}{(n+2)(3n^2+1)}-\frac{n(n+2)}{(n+2)(3n^2+1)}$$

$$\frac{3Cn^2+C-n^2-2n}{(n+2)(3n^2+1)}=\frac{(3C-1)n^2-2n+C}{3n^3+6n^2+n+2}$$

$$\sum_{n=3}^{\infty}\frac{(3C-1)n^2-2n+C}{3n^3+6n^2+n+2}$$

If $\boxed{C=\frac{1}{3}}$, then the limit comparison test with $\sum_{n=3}^{\infty}\frac{1}{n^2}$ shows that the series converges

If C does not equal $\frac{1}{3}$, then the limit comparison test with $\sum_{n=3}^{\infty}\frac{1}{n}$ shows that the series diverges

6. If $a_n=3+\frac{\ln(n)^6}{\sqrt{n}}$, then the sequence a_n:

(a) Converges to 3.
(b) Diverges to infinity.
(c) Diverges, but not to infinity.
(d) Converges to 6.
(e) Converges to 4.

$$a_n=3+\frac{\ln(n)^6}{\sqrt{n}}$$

$$\lim_{n\to\infty}3+\frac{\ln(n)^6}{\sqrt{n}}$$

For large n, $\ln(n)<n^c<c^n$

$$\frac{\ln(n)^6}{\sqrt{n}}<\frac{(n^{\frac{1}{24}})^6}{n^{\frac{1}{2}}}=\frac{1}{n^{\frac{1}{4}}}$$

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{4}}} = 0$$

Therefore,

$$\lim_{n\to\infty} \frac{\ln(n)^6}{\sqrt{n}}$$

must also converge to zero.

$$3 + \lim_{n\to\infty} \frac{\ln(n)^6}{\sqrt{n}} = 3 + 0 = \boxed{3}$$

7. Determine if the series converges of diverges.

$$\sum_{n=1}^{\infty} \frac{4n^3}{e^{n^4}}$$

For large n: $\ln(n) < n^c < c^n$

$\frac{4n^3}{e^{n^4}} > 0$, and $\frac{4n^3}{e^{n^4}} < \frac{4n^3}{e^n} < \frac{4n^3}{n^5} = 4(\frac{1}{n^2})$

$$\sum_{n=1}^{\infty} 4\left(\frac{1}{n^2}\right)$$

The above series converges because it is a p-series with p > 1

$\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges and $\frac{4}{n^2} > \frac{4n^3}{e^{n^4}}$, therefore $\sum_{n=1}^{\infty} e^{n^4}$ $\boxed{\text{converges}}$ by comparison test.

8. Determine if the following integral converges or diverges

$$\int_1^{\infty} \frac{x^3 + 12x - 2}{x^6 + 5x^5 + 3x^2 - 1} dx$$

You can use the limit comparison test to prove the behavior of the improper integral

$$f(x) = \frac{x^3 + 12x - 2}{x^6 + 5x^5 + 3x^2 - 1} dx$$

The denominator is bottom heavy by three more powers of x so we expect it to converge

$$g(x) = \frac{1}{x^3}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3 + 12x - 2}{x^6 + 5x^5 + 3x^2 - 1} * \frac{x^3}{1}$$

$$\lim_{x\to\infty} \frac{x^6 + 12x^4 - 2x^3}{x^6 + 5x^5 + 3x^2 - 1}$$

$$\lim_{x\to\infty} \frac{x^6 + 12x^4 - 2x^3}{x^6 + 5x^5 + 3x^2 - 1} * \frac{\frac{1}{x^6}}{\frac{1}{x^6}}$$

$$\lim_{x\to\infty} \frac{1 + \frac{12}{x^2} - \frac{2}{x^3}}{1 + \frac{5}{x} + \frac{3}{x^4} - \frac{1}{x^6}} = \frac{1}{1} = 1$$

$\int_1^\infty g(x)$ converges by p-test, therefore $\int_1^\infty f(x)$ also $\boxed{\text{converges}}$ by Limit Comparison Test

9. Which of the following expressions best represents the volume of the solid of revolution found by rotating the area between the curve $y = 1 + x^2 - 2x^4$ and the x-axis for x on the interval (0,1) around the y-axis?

(a) $2\pi \int_0^1 x\sqrt{1 + (2x - 8x^3)^2}dx$

(b) $2\pi \int_0^1 \sqrt{1 + (2x - 8x^3)^2}dx$

(c) $2\pi \int_0^1 (x + x^3 - 2x^5)dx$

(d) $2\pi \int_0^1 (1 + x^2 - 2x^4)dx$

(e) $2\pi \int_0^1 y\sqrt{1 + (2x - 8x^3)^2}dx$

We want to find the area between the curve $y = 1 + x^2 - 2x^4$ and x-axis for x values in the range of [0,1] around the y-axis

This can be most easily done by using the shell/cylinder method

$$\int_a^b 2\pi xy dx = \int_0^1 2\pi xy dx$$

$$\int_0^1 2\pi x(1 + x^2 - 2x^4)dx$$

$$\boxed{V = 2\pi \int_0^1 (x + 2x^3 - 2x^5)dx}$$

10. Determine the type of integral (proper, improper, type I, or type II), and point(s) that make the integral fall under that classification.

$$\int_2^3 \frac{\sin(x)}{x^2 - 3x + 2}dx$$

(a) Type I

(b) Proper

(c) Type II - integrand undefined at $x = 2$

(d) Type II - integrand undefined at $x = 2$ and $x = 3$

(e) Type II - integrand undefined at $x = 3$

This is a type II integral. The denominator is eliminated at $x = 1$ and $x = 2$. The answer is **(c)**.

11. Determine which of the following is true about the series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

(a) Not enough information to make a determination

(b) The series converges to $\frac{2}{5}$

(c) The series diverges

(d) The series converges absolutely

(e) The series converges only conditionally

The correct answer for this problem is (d). $\frac{1}{n^2}$ converges by the p-test. Since a_n is decreasing and $a_n \to 0$ as $n \to \infty$, the series $\boxed{\text{absolutely converges}}$.

12. Find the sum of the geometric series:

$$12 + 12\left(\frac{7}{19}\right) + 12\left(\frac{7}{19}\right)^2 + 12\left(\frac{7}{19}\right)^3 \ldots$$

(a) 20

(b) 8

(c) 19

(d) 14

(e) $\frac{7}{19}$

This is a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} |r| < 1$$

$r = \frac{7}{19}$, $a = 12 \to \frac{12}{1-\frac{7}{19}} = \boxed{19}$

13. Find the sum of the series:

$$\sum_{n=0}^{\infty} \frac{\cos^n(x)}{3^n}$$

(a) $\frac{3}{3-\cos(x)}$

(b) $\frac{1}{3-\cos(x)}$

(c) $\frac{2}{3-\cos(x)}$

(d) $\frac{3}{3-\sin(x)}$

(e) $\frac{1}{3-\sin(x)}$

Identify the geometric series

$$\sum_{n=0}^{\infty} ar^{n-1},\ r = \frac{(n+1)_{term}}{n_{term}} = \left[\frac{\cos^{n+1}(x)}{3^{n+1}}\right]\frac{3^n}{\cos^n(x)}$$

$$\frac{\cos^1(x)}{3} = \frac{1}{3}\cos(x) < 1$$

This converges to: $\frac{a}{1-r}$

$$a = 1, r = \frac{\cos(x)}{3} \to \boxed{\frac{3}{3-\cos(x)}}$$

14. Determine the value of the series, if it converges:

$$\sum_{n=1}^{\infty} \frac{5^{n+2}}{6^n}$$

(a) 64

(b) The series diverges

(c) 2

(d) 125

(e) 25

$$\sum_{n=1}^{\infty} \frac{5^{n+2}}{6^n} \to \left(\frac{5}{6}\right)5^2 \qquad r = \frac{5}{6}$$

$\frac{a}{1-r}$ where a is the first term.

$$a = \frac{5^3}{6}$$

$$\frac{5^3}{6}\left(\frac{1}{1-\frac{5}{6}}\right) = 5^3 = \boxed{125}$$

15. Does the series converge or diverge?

$$\sum_{n=1}^{\infty} e^{\frac{3}{n}}$$

Test for divergence

$$\lim_{n\to\infty} e^{\frac{3}{n}} \to 1$$

It $\boxed{\text{diverges}}$.