

Exam 3 Review Session

Math 231E



Please join the
queue for
attendance!



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Outline

1. Please join the queue
2. Mini review of some topics covered
3. Practice! → CARE Worksheet, Practice Exams
 - a. Please raise hands for questions rather than put them in the queue



Need extra help? → 4th Floor Grainger Library

Subject	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Math 231 (E)	4pm-10pm	1pm-5pm 8pm-10pm		1pm-5pm 8pm-10pm	6pm-8pm		2pm-4pm



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Improper Integrals

- Improper Integrals: FTC does not hold since functions are **not continuous along the interval of integration**.

- Type I: Infinite Interval

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$
$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

- Type II: Discontinuous Interval

$$\int_{a_b}^b f(x) dx = \lim_{t \rightarrow b^-} \int_{a_b}^t f(x) dx$$
$$\int_a^{a_b} f(x) dx = \lim_{t \rightarrow a^+} \int_t^{a_b} f(x) dx$$
$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Improper Integrals

- Comparison Principal: If given two functions $g(x)$ and $h(x)$ and we want to take the integral to infinity, **and we know that $g(x)$ is always smaller than $h(x)$** , then:
 - If $g(x)$ diverges, then $h(x)$ must as well.
 - If $h(x)$ converges, then $g(x)$ must as well.

Applications

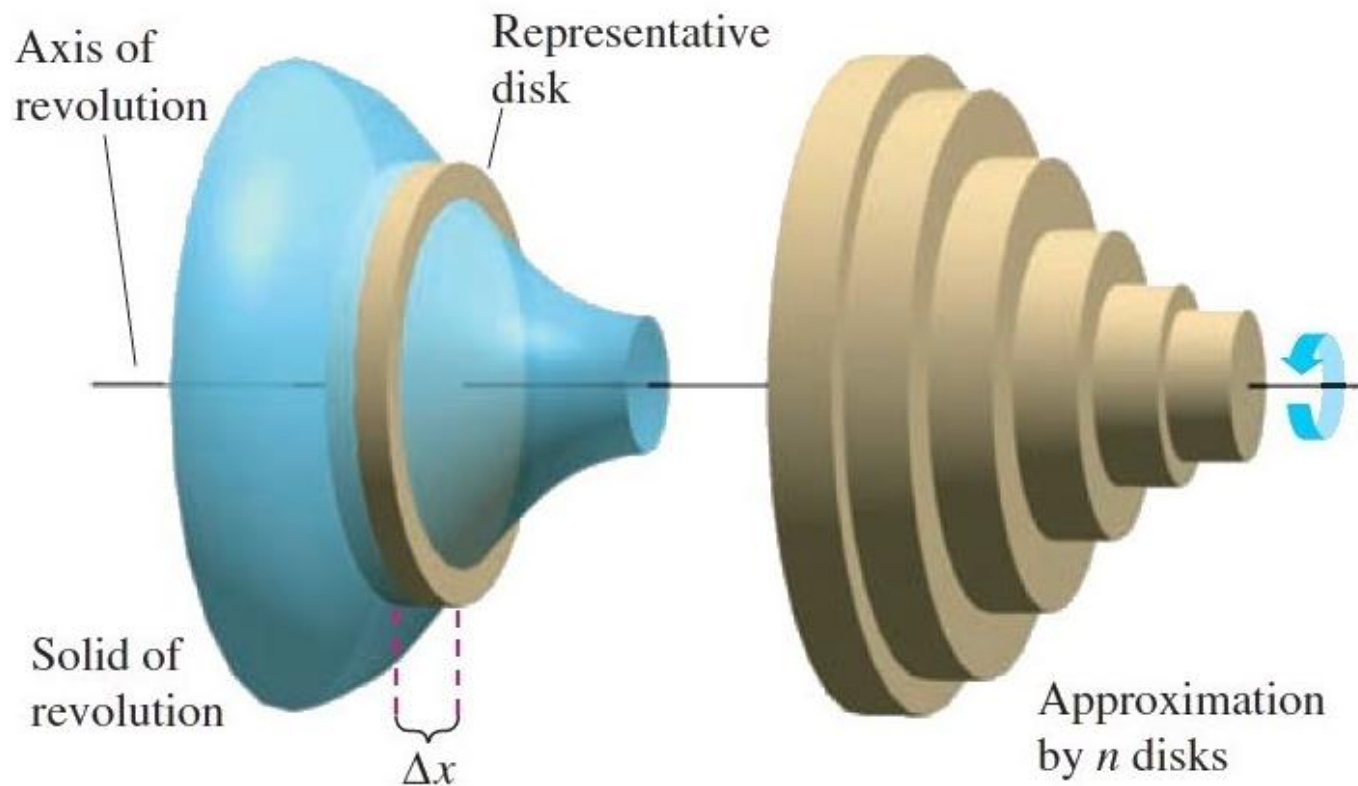
- Volume

Disk Method

- Using the formula for area of a circle
 - Putting it in the integral adds each circle in the bounds

$$V = \int_a^b \pi r^2 dx$$

Where the $r = f(x)$



Applications

- Volume

Washer Method

- Must subtract big function minus small function to find the in between region

(Top - Bottom)

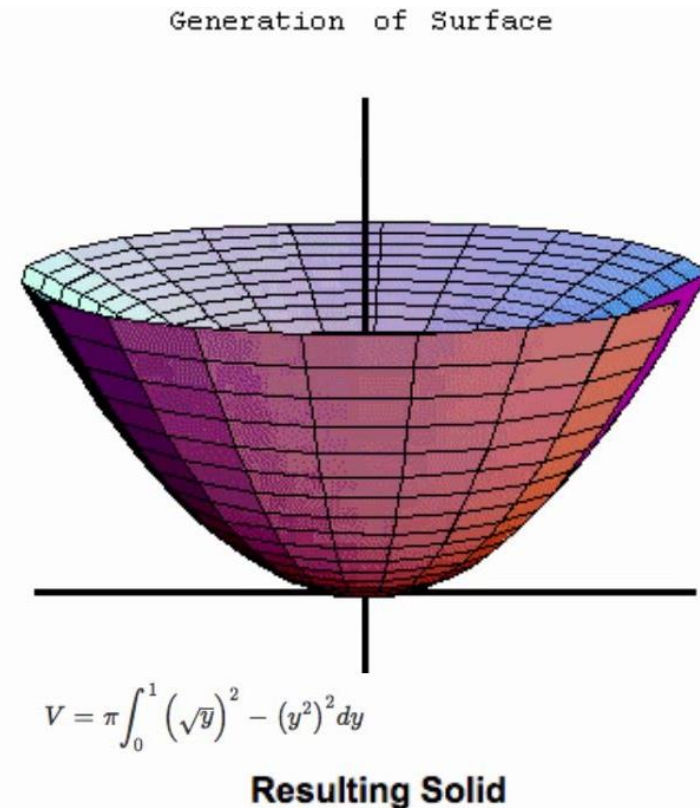
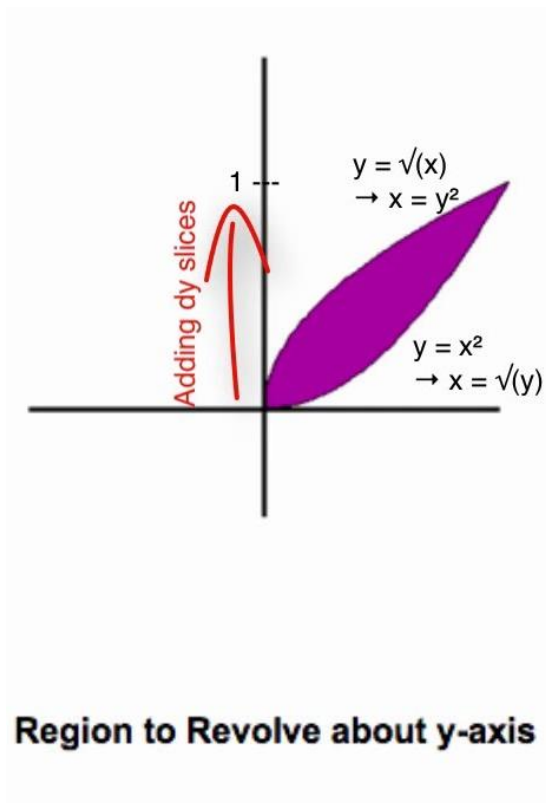
(Right - Left)

$$V = \int_a^b \pi(R^2 - r^2)dx$$

Where $R = f(x)$

$r = g(x)$

$f(x) > g(x)$



$$V = \pi \int_0^1 (\sqrt{y})^2 - (y^2)^2 dy$$



Applications

- Arclength (“height”):

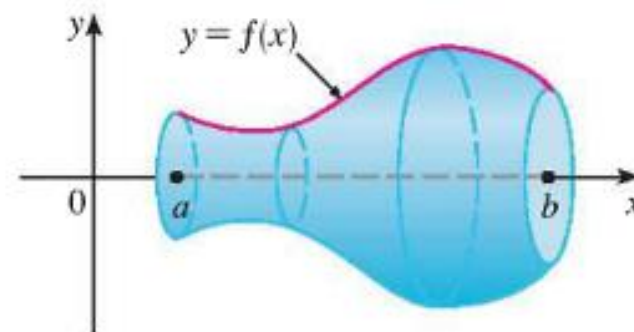
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \text{ if } y = f(x)$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \text{ if } x = h(y)$$

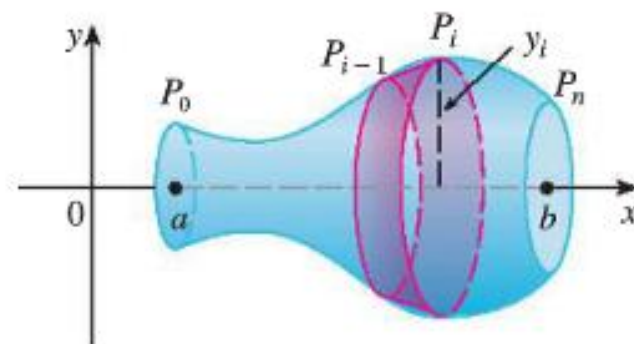
- Surface of Revolution:

$$SA = \int 2\pi y ds \text{ rotation about x-axis}$$

$$SA = \int 2\pi x ds \text{ rotation about y-axis}$$



(a) Surface of revolution



(b) Approximating band

Applications Example

How to set up the surface area equation when rotating $y = \sqrt{9 - x^2}$ about the y-axis?

- Because about y-axis, using $SA = \int 2\pi x ds$
 - We need x and ds

To get x, rearrange the given equation

$$\begin{aligned}y &= \sqrt{9 - x^2} \\y^2 &= 9 - x^2 \\x^2 &= 9 - y^2 \\x &= \sqrt{9 - y^2}\end{aligned}$$

Substituting...

$$SA = \int 2\pi \sqrt{9 - y^2} ds$$

To get ds, use the equation for ds that gives us a dy (since x is in terms of y)

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \text{ if } x = h(y)$$

Derivative of x...

$$\frac{dx}{dy} = \frac{-y}{\sqrt{9 - y^2}}$$

Plug back in...

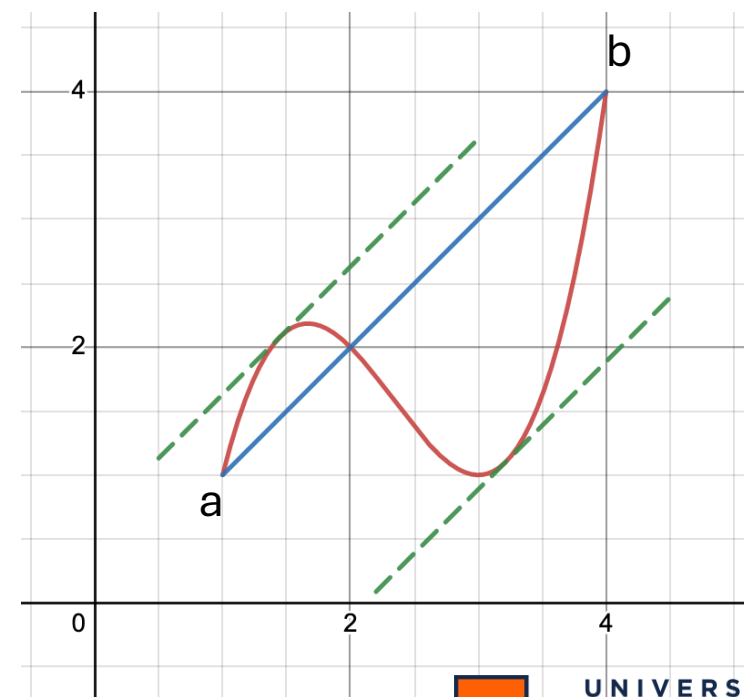
Work

- Work: Force over a distance

$$W = \int F(x) dx$$

- If the force is not constant.
- Average Value of a function over an interval

$$y_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$



Series / Sequences

- **Sequence:** Just the list of the numbers
 - Limits of sequences
 - Treat it like a function
 - Convergence
 - Treat it like a function
 - Derivative can tell you if it is always increasing or decreasing
- **Series:** The sum of a sequence
 - If a series converges, then the sequence must converge as well.
 - **However:** If sequence converges, then the series may or may not converge.
 - $\sum a_n$ converges if the limit of the series converges.

Integral Test

- Let $a_n = f(n)$:
 - $\int_k^\infty f(x) dx$ converges if the series converges ($\sum a_k$).

Must be:

- Continuous
- Positive
- Decreasing

1. If $\int_k^\infty f(x) dx$ is convergent so is $\sum_{n=k}^\infty a_n$.

2. If $\int_k^\infty f(x) dx$ is divergent so is $\sum_{n=k}^\infty a_n$.

- P-test:
 - The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Comparison/Limit Tests

- **Direct Comparison Test:**

- Let $0 \leq a_n \leq b_n$.
 - If the series of b_n converges, then the series of a_n does as well.
 - If the series of a_n diverges, then the series of b_n does as well.

- **Limit Comparison Test:**

- Let $0 \leq a_n, b_n$
 - If the limit of $a_n/b_n = C$, and C is a nonzero, finite number (ie. not zero or infinity)
 - Then one of two things:
 - Both a_n and b_n converge.
 - Both a_n and b_n diverge.

Alternating Series Test

- **What is an Alternating Series?**

- The series is changing signs with each subsequent term
- $\sum a_n (-1)^{n+1}$

- **Alternating Series Test**

- With series $\sum a_n$, $a_n = (-1)^n b_n$ OR $a_n = (-1)^{n+1} b_n$
 - If $\lim_{n \rightarrow \infty} b_n = 0$
- AND
- b_n is a decreasing sequence
- **The series $\sum a_n$ is convergent**

Root Test

Theorem 2.1 (Root Test). Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us define

$$b_n = \sqrt[n]{|a_n|} = |a_n|^{1/n},$$

and assume that $\lim_{n \rightarrow \infty} b_n = L$. Then

1. if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.



Ratio Test

Theorem 3.1. Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us assume that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then

1. if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.



Absolute Convergence

- **Absolute Convergence:**

- If the absolute value of a series, then the series is absolutely convergent.

- **Conditional Convergence:**

- If a series is convergent, but the absolute value of the series diverges, then the series is conditionally convergent.

- **Negative signs can only help convergence!**

Strategies

1. Check divergence with limit

2. Look for easy P-Test/Geometric

3. Inspection

TEST	SERIES	CONVERGES IF...	DIVERGES IF...	COMMENTS
n th Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$	n/a	$\lim_{n \rightarrow \infty} a_n \neq 0$	should be first test used. Inconclusive if limit = 0.
Geometric Series Test	$\sum_{n=1}^{\infty} a_n r^{n-1}$	$ r < 1$	$ r \geq 1$	use if there is a "common ratio" $S_n = \frac{a}{1-r}$
P-Series Test	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	harmonic series when $p=1$. Useful for comparison tests.
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(x)$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	$f(x)$ must be continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges	to show convergence, find a larger series. to show divergence, find a smaller series.
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ diverges	apply l'hospital's rule if necessary; inconclusive if limit equals 0 or ∞
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$a_{n+1} \leq a_n$, $\lim_{n \rightarrow \infty} a_n = 0$	$\lim_{n \rightarrow \infty} a_n \neq 0$	must prove that the limit equals 0 and that terms are decreasing
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1$	test fails if: $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } > 1$	test fails if: $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = 1$

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