Math 231
Midterm 3 Review

A Section: Bahreini

Series

- Series: The sum of a sequence.
  - If a series converges, then the sequence must converge as well.
  - **However:** If sequence converges, then the series may or may not converge.
  - Σa_n converges if the limit of the series converges.
- Geometric series:
  - $\Sigma ar^{k-1} = a + ar + ar^2 + ar^3 \dots .$
  - Will converge if $|r| < 1$
- Other techniques:
  - Evaluate the partial sums (first bit of sums) of a series and see how the series behaves
- If Σa_k converges, then $\lim_{x \to \infty} a_n = 0$

Integral Test

- Let an = f(n):
  - $\int f(x)dx$ (from k to infinity) converges if the series converges (Σa_k).

    Must be:
    - Continuous
    - Positive
    - Decreasing

1. If $\int_k^\infty f(x)\, dx$ is convergent so is $\sum_{n=k}^\infty a_n$.
2. If $\int_k^\infty f(x)\, dx$ is divergent so is $\sum_{n=k}^\infty a_n$.

- P-test:
  - The series $\Sigma \frac{1}{n^p}$ converges if p > 1 and diverges if p ≤ 1.

Comparison Tests

- **Direct Comparison Test:**
  - Let $0 \leq an \leq bn$.
    - If the series of bn converges, then the series of an does as well.
    - If the series of an diverges, then the series of bn does as well.
- **Limit Comparison Test:**
  - Let $0 \leq an, bn$
    - If the limit of $an/bn = C$, and C is a nonzero, finite number (ie. not zero or infinity)
    - Then one of two things:
      - Both an and bn converge.
      - Both an and bn diverge.

Alternating Series Test

- **What is an Alternating Series?**
  - The series is changing signs with each subsequent term
  - $\Sigma a_n(-1)^{n+1}$
- **Alternating Series Test**
  - With series Σa_n, $a_n = (-1)^n b_n$ OR $a_n = (-1)^{n+1} b_n$
    - If $\lim_{n\to\infty} b_n = 0$

      AND
    - b_n is a decreasing sequence
  - **The series Σa_n is convergent**

Absolute Convergence

- **Absolute Convergence:**
  - If the absolute value of a series, then the series is absolutely convergent.
- **Conditional Convergence:**
  - If a series if convergent, but the absolute value of the series diverges, then the series is conditionally convergent.
- **Negative signs can only help convergence!**

Root Test

Theorem 2.1 (Root Test). Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us define

$$b_n = \sqrt[n]{|a_n|} = |a_n|^{1/n},$$

and assume that $\lim_{n\to\infty} b_n = L$. Then

1. if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.

Ratio Test

Theorem 3.1. Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us assume that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then

1. if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.

Strategies

- Classify what kind of series you are working with
- Determine what kind of test to use

TEST	SERIES	CONVERGES IF...	DIVERGES IF...	COMMENTS
nth Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$	n/a	$\lim_{n\to\infty} \neq 0$	should be first test used. Inconclusive if limit = 0.
Geometric Series Test	$\sum_{n=1}^{\infty} a_n r^{n-1}$	$\|r\| < 1$	$\|r\| \geq 1$	use if there is a "common ratio" $S_n = \frac{a}{1-r}$
P-Series Test	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	harmonic series when p=1. Useful for comparison tests.
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(x)$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	$f(x)$ must be continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges	to show convergence, find a larger series. to show divergence, find a smaller series.
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ diverges	apply l'hospital's rule if necessary; inconclusive if limit equals 0 or ∞
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$a_{n+1} \leq a_n$, $\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	must prove that the limit equals 0 and that terms are decreasing
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$	test fails if : $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} < 1$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} > 1$	test fails if: $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = 1$

Power Series

- Can be defined by the form:

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

  - C_n are the coefficients
  - Series is a function of x

- Can be centered at any number:

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

Power Series

- Domain of Convergence: For what values will the series converge?
  - Use tests to find out what values of x satisfies convergence criteria.

Theorem 3.1. For any power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

there are exactly three possibilities for the domain of convergence (DOC) and radius of convergence (ROC).

1. Converges only at $x = a$, or
   DOC = $\{a\}$, ROC = 0;
2. Converges for all x, or
   DOC = $(-\infty, \infty))$, ROC = ∞;
3. There is an R such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$,
   ROC = R.

Remark 3.2. In the case with a radius of convergence R with $0 < R < \infty$, we have to check the endpoints "by hand".