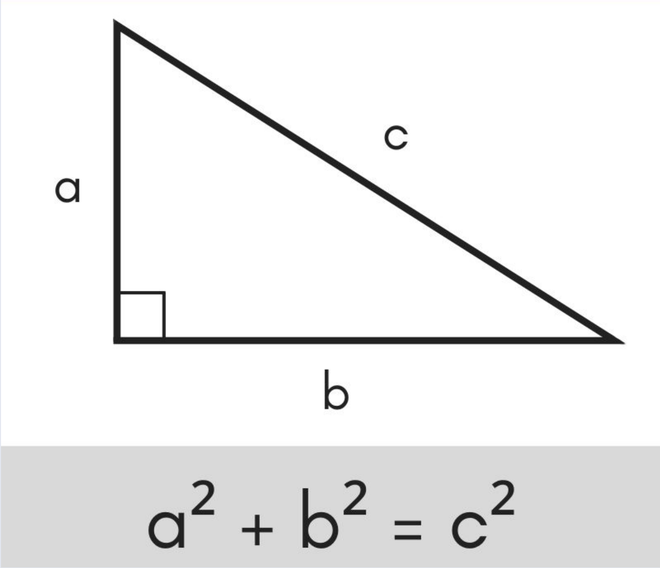
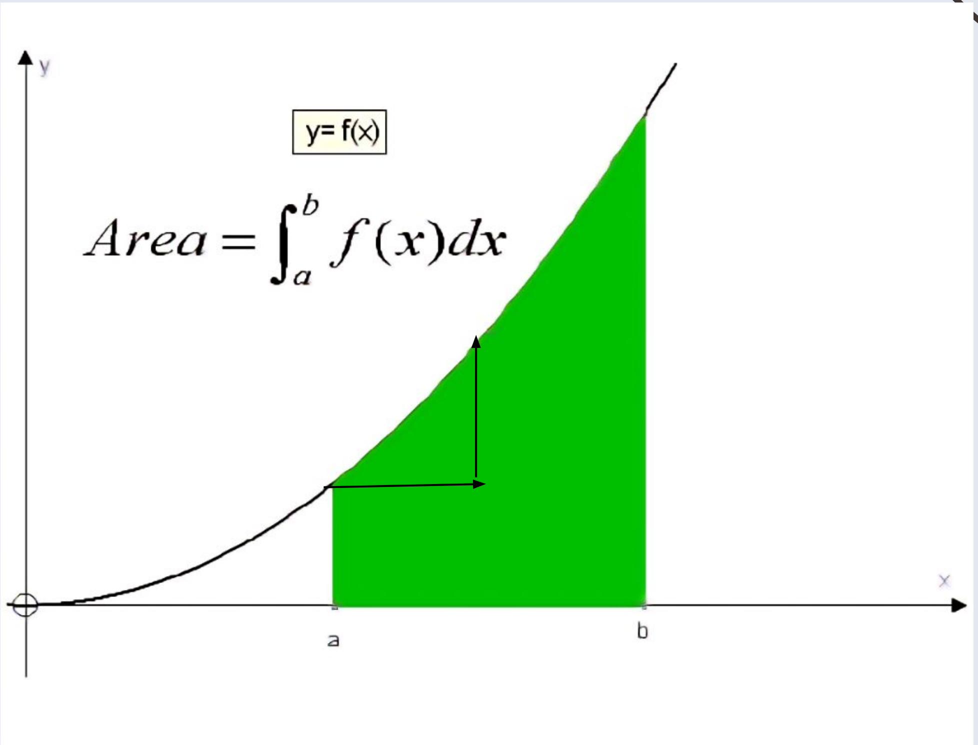
MATH 231 Exam Review

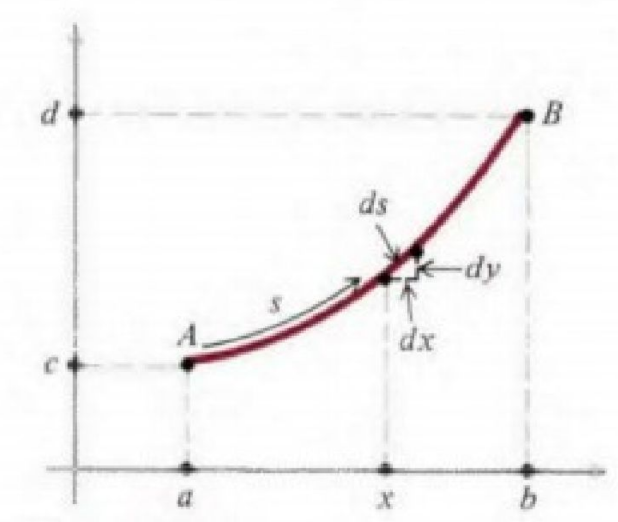
Midterm 02

Arc Length





Arc Length



$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

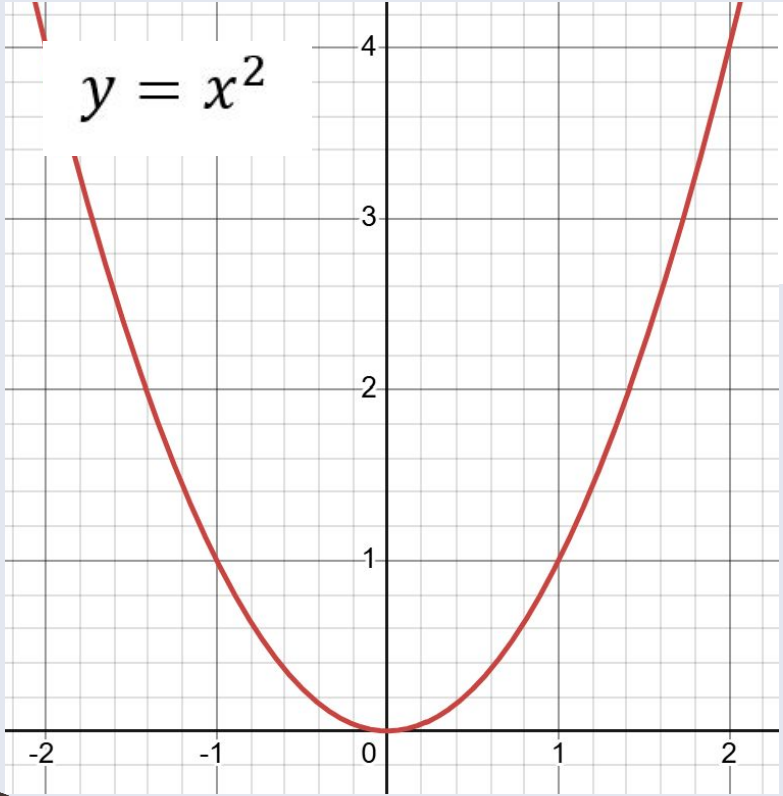
$$= \sqrt{\left(1 + \frac{dy^2}{dx^2}\right) dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$\boxed{\text{length of arc } AB = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

When to use which formula… and how to go about each one

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx \implies a \le x \le b \qquad y = x^2$$

$$L = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2}dy \implies a \le y \le b \qquad x = \sqrt{y}$$

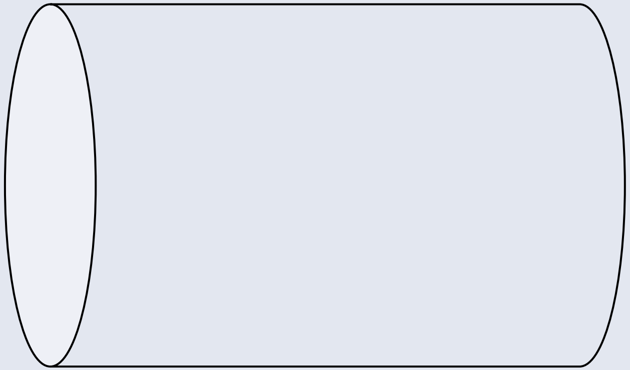


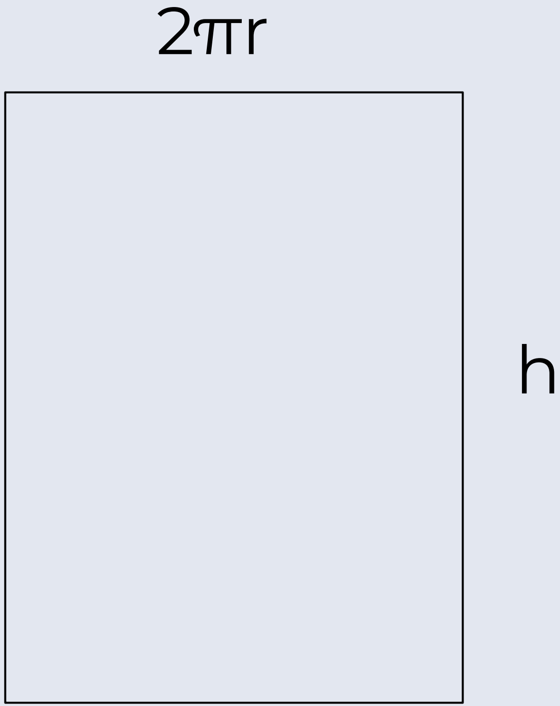
$$\int_0^2 \sqrt{1+(2x)^2}\,dx = \int_0^4 \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

General steps for solving arc length problems:

1. Write down formula that makes the most sense based on what you are given in the problem
2. Find the derivative
3. Set up the integrand and solve

Surface Area of a Revolution

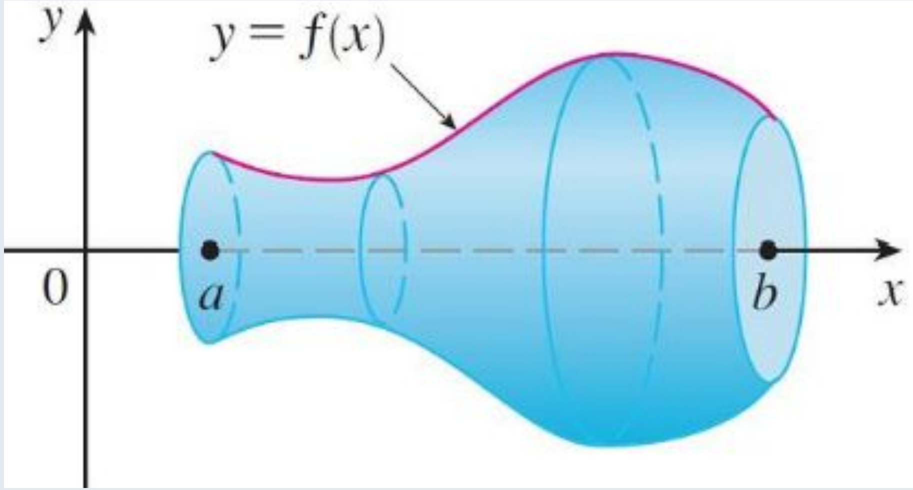




$$A = 2\pi r \times h$$

Surface Area of Revolution

We apply the same knowledge to more complex shapes, the arc length will be the ‘h’ and then the given function will be your circumference.



Surface Area of Revolution

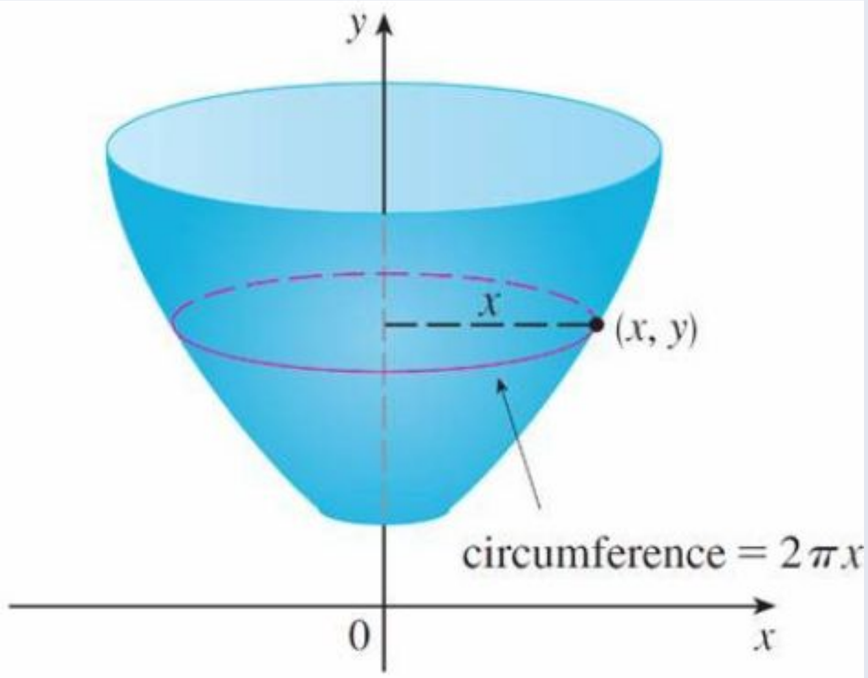
$$S = \int 2\pi y \, ds$$ ⟹ Around x-axis

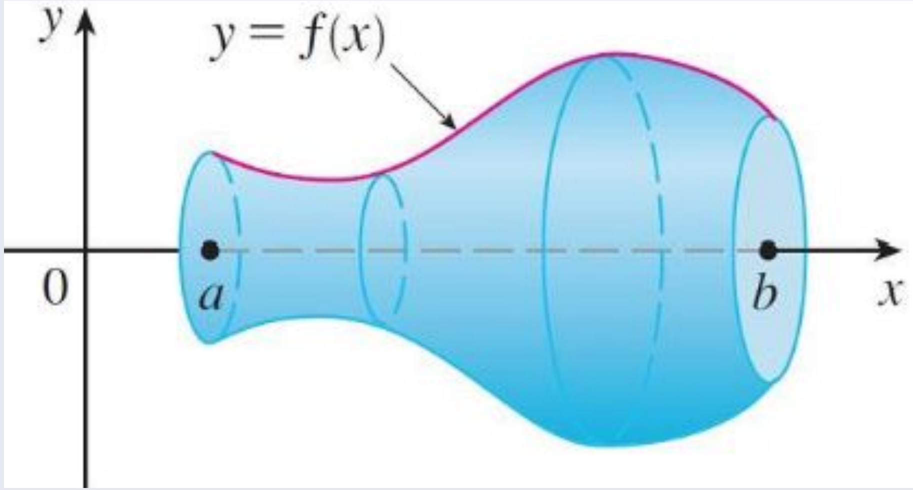
$$S = \int 2\pi x \, ds$$ ⟹ Around y-axis

$$ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$ds = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2} dy$$

Visualizing what is going on



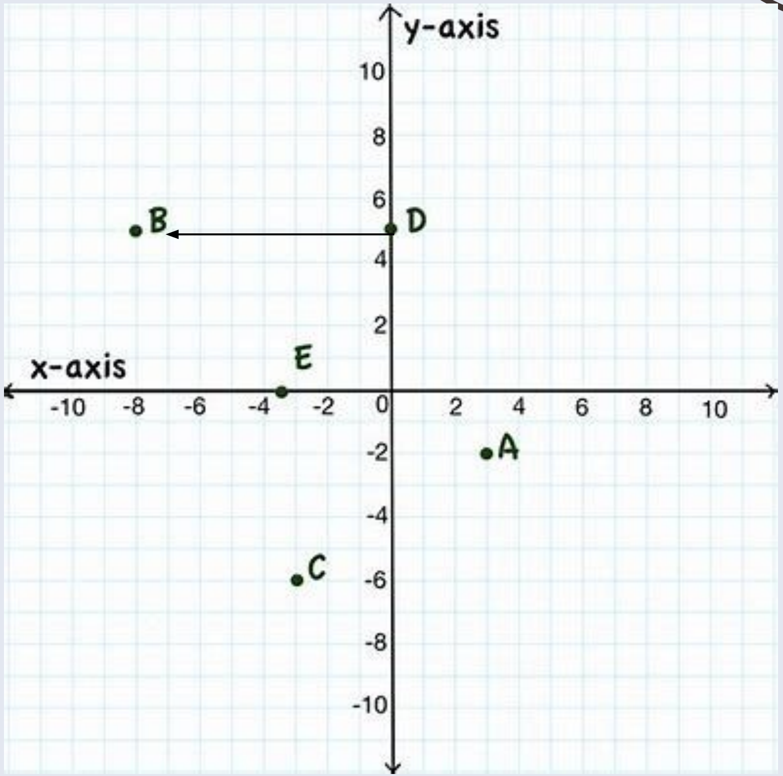


8.3

Center of mass: $$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Moments about axis: $$M_y = \sum m_i x_i$$ $$M_x = \sum m_i y_i$$

Center of mass coordinates: $$\bar{x} = \frac{M_y}{m}$$ $$\bar{y} = \frac{M_x}{m}$$



8.3 uniform density

$$\bar{x} = \frac{1}{A}\int_a^b x f(x) dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2 dx$$

If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, the centroid of $\mathcal{R}$ is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}dx$$

Sequences

Sequence: Just a list of the numbers

- Pattern Recognition!

- To test whether convergent or divergent take the limit, if the limit exists: convergent, if not! It is divergent.
  - If the sequence is a function, take the limit of the function
  - If cannot take limit, Squeeze Theorem!

Root Test

Theorem 2.1 (Root Test). Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us define

$$b_n = \sqrt[n]{|a_n|} = |a_n|^{1/n},$$

and assume that $\lim_{n\to\infty} b_n = L$. Then

1. if $L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.

Ratio Test

Theorem 3.1. Let a_n be a sequence and $\sum_{n=1}^{\infty} a_n$ be the associated series. Let us assume that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then

1. if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely;
2. if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges;
3. if $L = 1$, then no information is obtained.

Useful Squeeze Theorem

- $\lim_{x\to\infty}\left(\frac{\sin(x)}{x}\right) = 0$
- $\lim_{x\to0}\left(\frac{\sin(ax)}{x}\right) = a$
- $\lim_{x\to0}\left(\frac{\sin(x)}{x}\right) = 1$
- $\lim_{x\to0}\left(\frac{\cos(x)-1}{x}\right) = 0$

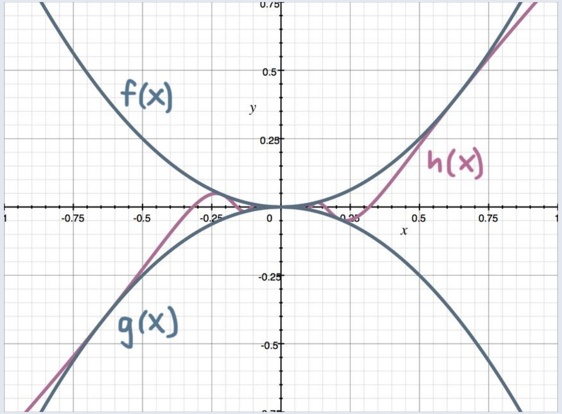
If $f(x) \le h(x) \le g(x)$

and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = L$$

then

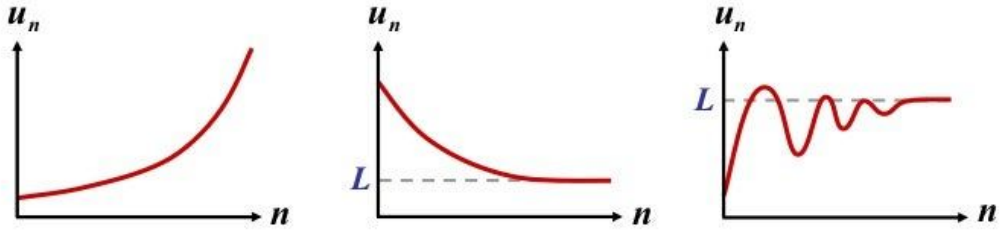
$$\lim_{x\to c} h(x) = L$$



Sequences

Convergence and Divergence.

Some sequences never stop increasing, while others eventually settle at a particular number.



If the numbers in a sequence continue to get further and further apart, the sequence diverges.

If a sequence tends towards a limit, it is described as convergent.

Sequence Convergence

- Convergence:
  - Increasing
    - if all $a_n < a_{n+1}$
  - Decreasing
    - if all $a_n > a_{n+1}$
  - Bounded from Below
    - If there existed a number m such that $\mathrm{m} \leq a_{n+1}$
  - Bounded from Above
    - If there existed a number M such that $\mathrm{M} \geq a_{n+1}$
  - If a sequence is bounded (from above and below) and monotonic (increasing or decreasing only), then it is convergent
    - If is not both of these, does not necessarily mean it is divergent

Series

- Series: The sum of a sequence.
  - If a series converges, then the sequence must converge as well.
  - **However:** If sequence converges, then the series may or may not converge.
  - Σa_n converges if the limit of the series converges.
- Geometric series:
  - $\Sigma ar^{k-1} = a + ar + ar^2 + ar^3 \ldots.$
  - Will converge if $|r| < 1$
- Other techniques:
  - Evaluate the partial sums (first bit of sums) of a series and see how the series behaves
- If Σa_k converges, then $\lim_{x \to \infty} a_n = 0$

Strategies

1. Check divergence
2. Look for easy P-Test/Geometric
3. Inspection

TEST	SERIES	CONVERGES IF…	DIVERGES IF…	COMMENTS
nth Term Test for Divergence	$\sum_{n=1}^{\infty} a_n$	n/a	$\lim_{n\to\infty} \neq 0$	should be first test used. Inconclusive if limit = 0.
Geometric Series Test	$\sum_{n=1}^{\infty} a_n r^{n-1}$	$\|r\| < 1$	$\|r\| \geq 1$	use if there is a "common ratio" $S_n = \frac{a}{1-r}$
P-Series Test	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	harmonic series when p=1. Useful for comparison tests.
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(x)$	$\int_1^{\infty} f(x)\,dx$ converges	$\int_1^{\infty} f(x)\,dx$ diverges	$f(x)$ must be continuous, positive, and decreasing
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$	$0 \leq a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$, $\sum_{n=1}^{\infty} b_n$ diverges	to show convergence, find a larger series. to show divergence, find a smaller series.
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\sum_{n=1}^{\infty} b_n$ diverges	apply l'hospital's rule if necessary; inconclusive if limit equals 0 or ∞
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$a_{n+1} \leq a_n$, $\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	must prove that the limit equals 0 and that terms are decreasing
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$	test fails if : $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} < 1$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} > 1$	test fails if: $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = 1$