



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 231 — Engineering Calculus

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Oct. 20, 7-8:30pm Grace and Sushrut

Session 2: Oct. 21, 7-8:30pm Sofi and Bella

Can't make it to a session? Here's our schedule by course:

<https://care.grainger.illinois.edu/tutoring/schedule-by-subject>

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: <https://queue.illinois.edu/q/queue/844>
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Evaluate the following integral:

$$\int e^x \cos(x) dx$$

(a) $\frac{e^x}{4}(\sin(x) + \cos(x)) + C$

(b) $\frac{e^x}{2}(\sin(x) + \cos(x)) + C$

(c) $\frac{e^x}{2}(\sin(x)) + C$

(d) $\frac{e^x}{6}(\sin(x) + \cos(x)) + C$

Call $\int e^x \cos(x) dx = I$. Use integration by parts

$$\begin{aligned} v &= e^x \\ dv &= e^x dx \end{aligned}$$

$$u = \cos(x) dx$$

$$du = -\sin(x)$$

$$I = e^x \cos(x) - \int e^x - \sin(x) dx$$

Use integration by parts again

$$\begin{aligned} v &= e^x \\ dv &= e^x dx \end{aligned}$$

$$u = -\sin(x) dx$$

$$du = -\cos(x)$$

$$I = e^x \cos(x) - [-e^x \sin(x) - \int -e^x \cos(x)] dx$$

$$I = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) dx$$

$$I = e^x(\sin(x) + \cos(x)) - I$$

$$2I = e^x(\sin(x) + \cos(x)) \rightarrow \boxed{I = \frac{e^x}{2}(\sin(x) + \cos(x)) + C}$$

2. What is the best substitution to make in order to solve the following integral?

$$\int y^2 \sqrt{y^2 + 4} dy$$

- (a) $y = \ln(\theta)$
- (b) $y = 2 \sec(\theta)$
- (c) $y = \sin(\theta)$
- (d) $y = 2 \tan(\theta)$
- (e) $y = e^t + 1$

We want to simplify the radical. Recalling that

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

the substitution $y = 2 \tan(\theta)$ and $dy = 2 \sec^2(\theta)$ yields

$$\int 4 \tan^2(\theta) \sqrt{4(\tan^2(\theta)) + 4} (2 \sec^2(\theta)) d\theta$$

$$\int 8 \tan^2(\theta) \sec^2(\theta) \sqrt{4(\tan^2(\theta) + 1)} d\theta$$

$$\int 16 \tan^2(\theta) \sec^3(\theta) d\theta$$

3. Evaluate the following integral:

$$\int_1^{e^3} 4x^3 \ln(x)$$

- (a) $\frac{1+e^8}{2}$
- (b) $\frac{11e^{12}+1}{4}$
- (c) $3 + e^4$
- (d) $\frac{1+e^{12}}{2}$
- (e) $\frac{7e^8+1}{4}$

We will need to use integration by parts to solve this problem

$$u = \ln(x)$$

$$du = \frac{1}{x} dx$$

$$dv = 4x^3 dx$$

$$v = x^4$$

$$\int_1^{e^3} 4x^3 \ln(x) dx = x^4 \ln(x) - \int_1^{e^3} \frac{x^4}{x} dx$$

$$(3e^{12} - 0) - \frac{x^4}{4} \Big|_1^{e^3} = \boxed{\frac{11e^{12} + 1}{4}}$$

4. Evaluate the following indefinite integral

$$\int x\sqrt{x^2 + 2} dx$$

- (a) $\frac{1}{2}x^2\sqrt{x^2 + 2} + C$
- (b) $\frac{2}{3}(x^2 + 2)^{\frac{3}{2}} + C$
- (c) $\frac{1}{3}(x^2 + 2)^{\frac{3}{2}} + C$
- (d) $(x^2 + 2)^{\frac{3}{2}} + C$
- (e) $\frac{2}{3}x(x^2 + 2)^{\frac{3}{2}} + C$

We will need to use the substitution: $u = x^2 + 2, du = 2x dx$

$$\int x\sqrt{x^2 + 2} dx = \frac{1}{2} \int (2x)\sqrt{x^2 + 2} dx$$

$$\frac{1}{2} \int (2x)\sqrt{x^2 + 2} dx = \frac{1}{2} \int \sqrt{u} du$$

$$\frac{1}{2} \left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$\frac{u^{\frac{3}{2}}}{3} + C = \boxed{\frac{1}{3}(x^2 + 2)^{\frac{3}{2}} + C}$$

5. Let $y = f(x)$ be defined implicitly by $y^3 - xy = 10$, Find the $\frac{dy}{dx}$ in terms of y and x

- (a) $\frac{dy}{dx} = \frac{10y^2}{x^3}$
- (b) $\frac{dy}{dx} = \frac{10}{y^2 - x}$
- (c) $\frac{dy}{dx} = \frac{y}{3y^2 - x}$
- (d) $\frac{dy}{dx} = \frac{10}{3y^2} + \frac{x}{y}$

$$y^3 - xy = 10$$

$$3y^2 \frac{dy}{dx} - x \frac{dy}{dx} - y = 0$$

$$\boxed{\frac{dy}{dx} = \frac{y}{3y^2 - x}}$$

6. Suppose that $f(x)$ is continuous $[0, 1]$ and differentiable on $[0, 1]$ with $f(0) = 2$ and $f(1) = 5$. Exactly one of the following statements is true.
- (a) There exists a point c on $(\frac{1}{2}, 1)$ such that $f'(c) = 3$.
 - (b) None of the other answers are true.
 - (c) There exists a point c on $(0, 2)$ such that $f(c) = 0$.
 - (d) There exists a point c on $(0, 1)$ such that $f'(c) = 3$
 - (e) There exists a point c on $(2, 5)$ such that $f(\frac{1}{2}) = c$.

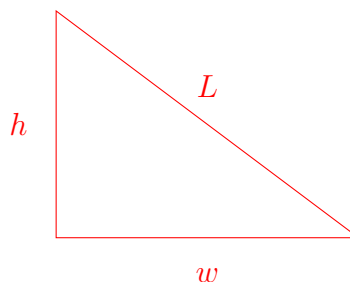
This is a simple mean value question

If $f(0) = 2$ and $f(1) = 5$, then there exists a C with $f'(C) = 3$.

The answer is **(d)**. None of the other options are necessarily true.

7. A 4m long ladder is propped up against a wall. The ladder begins to slip. At time $t = 3$ s, the base of the ladder is 2 m from the wall and moving away from the wall at 3 m/s. How fast is the top of the ladder moving vertically along the wall?
- (a) $-\sqrt{3}$ m/s
 - (b) -2 m/s
 - (c) $-3\sqrt{2}$ m/s
 - (d) $-\frac{13}{4}$ m/s
 - (e) $-3\sqrt{3}$ m/s

Let us denote the height of the ladder's contact with the wall as h , the distance from the wall to the base of the ladder as w and the length of the ladder is fixed at $L = 4$



Then we know from Pythagorean's Theorem that: $w^2 + h^2 = L^2$

Differentiating gives us:

$$2w \frac{dw}{dt} + 2h \frac{dh}{dt} = 0$$

Given that $w = 2$ and $\frac{dw}{dt} = 3$, we need to find h

$$h = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$$

Now, we can plug in and obtain:

$$2(3) + 2\sqrt{3} \frac{dh}{dt} = 0$$

$$\frac{dh}{dt} = \boxed{-\sqrt{3}}$$

8. We want to evaluate

$$\int_0^2 x^2 dx$$

using a Riemann sum of $n = 3$ terms. Let us define L_3 as the Riemann Sum if we choose the left endpoints, and R_3 if we choose the right endpoints. Then:

- (a) $L_3 = \frac{9}{32}, R_3 = \frac{15}{32}$
- (b) $L_3 = \frac{40}{27}, R_3 = \frac{112}{27}$
- (c) $L_3 = \frac{15}{81}, R_3 = \frac{41}{81}$
- (d) $L_3 = \frac{7}{32}, R_3 = \frac{55}{36}$
- (e) $L_3 = \frac{53}{36}, R_3 = \frac{55}{36}$

We have $\Delta x = \frac{2}{3}$

Choose the left-hand end points

$$x_1 = 0, x_2 = \frac{2}{3}, x_3 = \frac{4}{3}$$

$$L_3 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$0^2 \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 \left(\frac{2}{3}\right) + \left(\frac{4}{3}\right)^2 \left(\frac{2}{3}\right)$$

$$\boxed{L_3 = \frac{40}{27}}$$

Now, choose the right endpoints

$$x_1 = \frac{2}{3}, x_2 = \frac{4}{3}, x_3 = 2$$

$$R_3 = f(x_1)\Delta x + f(x_2)\delta x + f(x_3)\Delta x$$

$$\left(\frac{2}{3}\right)^2 \left(\frac{2}{3}\right) + \left(\frac{4}{3}\right)^2 \left(\frac{2}{3}\right) + (2)^2 \left(\frac{2}{3}\right)$$

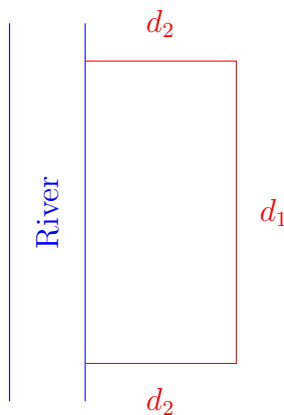
$$R_3 = \frac{112}{27}$$

9. A Farmer wants to build a rectangular field next to a river. The farmer will use the river as one side of the rectangle, but must build the other three sides of the rectangle with fence. There is a total amount of 120 meters of fence available. What is the largest possible area that can be enclosed?

- (a) 2400 m²
- (b) 1600 m²
- (c) 1800 m²
- (d) 800 m²

Let d_1 = the length of the side parallel to the river

Let d_2 = the length of the side perpendicular to the river



$$d_1 + 2d_2 = 120 \text{ or } d_2 = 60 - \frac{d_1}{2}$$

$$A = d_1 d_2 = d_1 \left(60 - \frac{d_1}{2}\right) = 60d_1 - \frac{(d_1)^2}{2}$$

Now, find the critical points

$$\frac{d}{d_1 x} (60d_1 - \frac{(d_1)^2}{2}) = 60 - d_1 = 0$$

$$d_1 = 60$$

$$60(30) = \boxed{1800 \text{ m}^2}$$

10. Let us define $f(x)$ by:

$$\int_{-3}^{x^2 \cos(x)} e^{-t^2} dt$$

Compute $f'(x)$

(a) $2x \cos(x) e^{-x^4 \cos^2(x)} - x^2 \sin(x) e^{-x^4 \cos^2(x)}$

(b) $x \cos(x) e^{-x^4 \cos^2(x)} + 2x^2 \sin(x) e^{-x^4 \cos^2(x)}$

(c) $e^{-x^4 \cos^2(x)} - e^{-9}$

(d) $2x \cos(x) e^{-x^2 \cos^2(x)} - x^2 \sin(x) e^{-x^2 \cos^2(x)}$

(e) $e^{x^4 \cos^2(x)}$

If we write: $h(x) = \int_{-3}^x e^{-t^2} dt$, then the fundamental theorem of calculus tells us:

$$h'(x) = e^{-x^2} \text{ where } f(h(x))h'(x)$$

We then have $f(x) = h(x^2 \cos(x))$, so:

$$f'(x) = h(x^2 \cos(x)) * \frac{d}{dx}(x^2 \cos(x))$$

$$f'(x) = h(x^2 \cos(x))(2x \cos(x) - x^2 \sin(x))$$

$$f'(x) = e^{-x^4 \cos^2(x)}(2x \cos(x) - x^2 \sin(x)) = \boxed{2x \cos(x) e^{-x^4 \cos^2(x)} - x^2 \sin(x) e^{-x^4 \cos^2(x)}}$$

11. Evaluate the following integral

$$\int \frac{5x - 1}{(x - 2)^2(x + 1)}$$

(a) $\ln|x - 2| + \frac{4}{2 - x} - 2 \ln|x + 1| + C$

(b) $\frac{1}{3} \ln|x - 4| + \frac{3}{1 - x} - \frac{2}{3} \ln|x + 3| + C$

(c) $\ln|x - 2| + \frac{3}{2 - x} - \ln|x + 1| + C$

- (d) $\frac{2}{3} \ln |x - 2| + \frac{3}{2-x} - \frac{2}{3} \ln |x + 1| + C$
 (e) $\frac{1}{4} \ln |x - 2| + \frac{3}{1-x} - \frac{2}{3} \ln |x + 1| + C$

First, decompose the given problem using partial fraction decomposition. Recall that there will be two terms for the squared term

$$\frac{5x - 1}{(x - 2)^2(x + 1)} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{x + 1}$$

$$5x - 1 = A(x - 2)(x + 1) + B(x + 1) + C(x - 2)^2$$

$$5x - 1 = Ax^2 - Ax - 2A + Bx + B + Cx^2 - 4Cx + 4C$$

$$5x - 1 = x^2(A + C) + x(-A + B - 4C) + (-2A + B + 4C)$$

From this we can get our three equations with three unknowns and solve:

$$(I) \ A + C = 0 \qquad (II) \ -A + B - 4C = 5 \qquad (III) \ -2A + B + 4C = -1$$

$$9C = -6$$

$$C = -\frac{2}{3}$$

$$A = -C = \frac{2}{3}$$

$$B = 5 + 4C + A = 5 - \frac{8}{3} + \frac{2}{3} = 3$$

$$\begin{aligned} \int \frac{5x - 1}{(x - 2)^2(x + 1)} dx &= \int \frac{\frac{2}{3}}{x - 2} + \frac{3}{(x - 2)^2} - \frac{\frac{2}{3}}{x + 1} dx \\ &= \boxed{\frac{2}{3} \ln |x - 2| + \frac{3}{2 - x} - \frac{2}{3} \ln |x + 1| + C} \end{aligned}$$

12. Evaluate the following integral

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$$

- (a) $\ln |x| + \ln(x^2 + 4) - \frac{1}{4} \arctan\left(\frac{x}{2}\right) + C$
 (b) $\ln |x| + \frac{1}{2} \ln(x^2 + 4) - \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C$
 (c) $\ln |x| + \frac{1}{2} \ln(x^2 + 8) - \frac{1}{2} \arctan\left(\frac{x}{4}\right) + C$
 (d) $\frac{1}{2} \ln |x| + \frac{1}{2} \ln(x^2 + 4) - \frac{1}{3} \arctan\left(\frac{x}{2}\right) + C$
 (e) $\frac{1}{2} \ln |x| + \frac{1}{2} \ln(x^2 + 2) - \frac{1}{3} \arctan\left(\frac{x}{4}\right) + C$

$$x^3 + 4x = x(x^2 + 4x)$$

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = (A + B)x^2 + Cx + 4A$$

$$A + B = 2, C = -1, \text{ and } 4A = 4$$

$$A = 1, B = 1, C = -1$$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} dx = \int \frac{1}{x} + \frac{x - 1}{x^2 + 4} dx$$

$$\int \frac{1}{x} + \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4} dx = \boxed{\ln|x| + \frac{1}{2} \ln(x^2 + 4) - \frac{1}{2} \arctan\left(\frac{x}{2}\right) + C}$$

Note: For the second term, use U-sub where $u = x^2 + 4$. For the third term in the integral, use trig integral for tangent

13. Evaluate the following integral

$$\int_0^4 \frac{dx}{(x^2 + 16)^{\frac{3}{2}}}$$

- (a) $\frac{1}{8\sqrt{2}}$
- (b) $\frac{1}{16\sqrt{3}}$
- (c) $\frac{1}{16\sqrt{2}}$
- (d) $\frac{1}{\sqrt{2}}$
- (e) $\frac{1}{8\sqrt{3}}$

Note that this is a perfect integral to use TRIG-SUB to solve. Use the following substitution in order to solve the problem:

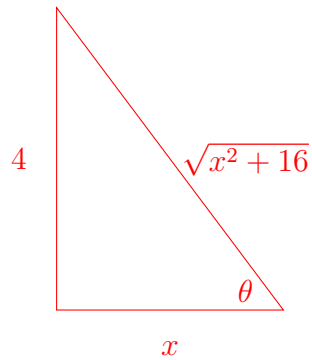
$$x = 4 \tan(\theta) \text{ and } dx = 4 \sec^2(\theta)$$

$$\int_0^4 \frac{4 \sec^2(\theta)}{4^3 \sec^3(\theta)} d\theta$$

$$\frac{1}{16} \int_0^4 \frac{1}{\sec(\theta)} d\theta$$

$$\frac{1}{16} \int_0^4 \cos(\theta) d\theta$$

$$\frac{1}{16} \sin(\theta) \Big|_0^4$$



$$\frac{1}{16} \frac{x}{\sqrt{x^2 + 16}} \Big|_0^4$$

$$\frac{1}{16} \left(\frac{4}{\sqrt{32}} - 0 \right) = \boxed{\frac{1}{16\sqrt{2}}}$$