Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Math 241 – Calculus III

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Oct. 18, 4-5:30 pm Pallab, Camila, Liz

Session 2: Oct. 19, 4-5:30 pm Meredith, Rick, Johail

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/845
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Consider the vector function $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$. Find the length from $(2,0,0)$ to $(2,0,4\pi)$ of the vector function.

   Need to figure out the values of t at start and end point

   a: $(2,0,0) \rightarrow$ z = 0 = t $\rightarrow$ t = 0

   b: $(2,0,4\pi)$ $\rightarrow$ z = 4π = t $\rightarrow$ t = 4π

   $L = \int_0^{4\pi} \sqrt{(-2sint)^2 + (2cost)^2 + 1^2} dt = \int_0^{4\pi} \sqrt{4sin^2 t + 4cos^2 t + 1} dt = \int_0^{4\pi} \sqrt{5} dt = 4\sqrt{5}\pi$

2. A tiny spaceship is orbiting a path given by $x^2 + y^2 = 4$. The solar radiation at a point (x, y) in the plane of the orbit is $f(x, y) = xy + 2y$.

   Use the method of *Lagrange multipliers* to find the maximum value and minimum value of solar radiation experienced by the tiny spaceship in its orbit.

   The function to be maximized/minimized is f, subject to the constraint $x^2 + y^2 = 4$, which we will call g.

$$\nabla f = \langle y, x+2\rangle \qquad \nabla g = \langle 2x, 2y\rangle$$

   We have 3 equations to work with. We know

$$\nabla f = \lambda \nabla g$$

   gives the following system of equations

$$\begin{gathered} y = \lambda(2x) \\ x + 2 = \lambda(2y) \\ x^2 + y^2 = 4 \end{gathered}$$

   Where the last equation is the constraint. We solve for λ in the first two equations:

$$\frac{y}{2x} = \lambda \text{ and } \frac{x+2}{2y} = \lambda$$

   Set them equal to each other to get an equation with just x and y:

$$\begin{gathered} \frac{y}{2x} = \frac{x+2}{2y} \\ 2y^2 = 2x^2 + 4x \end{gathered}$$

$$y^2 = x^2 + 2x$$

Plug this in to the constraint

$$x^2 + y^2 = 4$$

$$x^2 + x^2 + 2x = 4$$

$$\text{We get } x = 1,\ -2$$

Obtain the corresponding values for y:
For $x = 1$

$$y^2 = x^2 + 2x \qquad y = -\sqrt{3},\ \sqrt{3}$$

For $x = -2$

$$y^2 = x^2 + 2x \qquad y = 0$$

Now we must test whether these are minimum or maximum points. We have three points to test in our radiation function $f(x, y) = xy + 2y$

$$f(1, \sqrt{3}) = 3\sqrt{3}$$

$$f(1, -\sqrt{3}) = -3\sqrt{3}$$

$$f(-2, 0) = 0$$

$$\boxed{\text{Max: } 3\sqrt{3}}$$

$$\boxed{\text{Min: } -3\sqrt{3}}$$

3. Let $f(x, y)$ be a differentiable function on the disk $\{D : x^2 + y^2 \leq 400\}$, where:

(I) $f(x, y) = 19$ for every point on the boundary of the disk $x^2 + y^2 = 400$

(II) $f(0, 0) = 7$

(III) $f(x, y)$ has only one critical point which is at $(-1, 2)$

Decide which statement is true:

A) $f(-1, 2) > 7$

B) $f(-1, 2) < 7$

C) $f(-1, 2) = 7$

D) Not enough information is given

From the extreme value theorem we know that the maximum and minimum of the function in the domain D must (if they exist) be at the critical points or some points on the boundary.

Suppose $f(-1, 2) \geq 7$, then the minimum of the function is not at the point $(-1, 2)$ since $f(0, 0)$ has a smaller value. This would imply another critical point below $f(-1, 2)$, but the function only has one critical point.

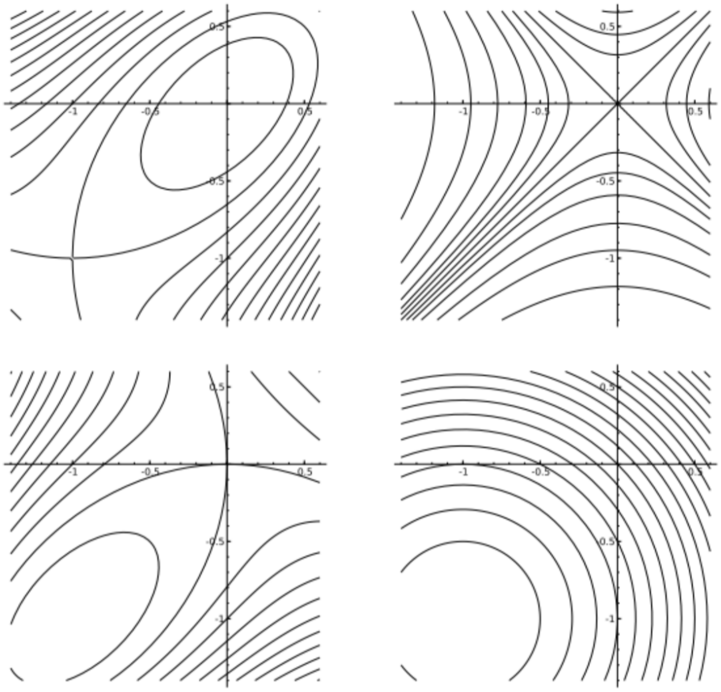
The minimum is also not on the boundary since $f(x, y) = 19 > 7$ for every point on the boundary of the disk.

Lastly, suppose $f(-1, 2) = 7$, then this implies that $(0, 0)$ is also a critical point, which disagrees with the condition. So the only way to satisfy the condition that there is one critical point is make $f(-1, 2) < 7$. This means it is below the origin.

4. Consider the function $f(x, y) = x^3 + y^3 + 3xy$

(a) The critical points of f are $(0, 0)$ and $(-1, -1)$. Classify them into local minima, local maxima and/or saddle points

(b) Based on your answer in (a), identify the correct contour diagram of f



(a) $f_x = 3x^2 + 3y$ $\qquad$ $f_y = 3y^2 + 3x$

$f_{xx} = 6x$ $\qquad$ $f_{yy} = 6y$

$$f_{xy} = f_{yx} = 3$$

$$\text{At } (0, 0)\ D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = -9 < 0 \to \text{SADDLE}$$

$$\text{At } (-1, -1)\ D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} = 36 - 9 = 27$$

$$27 > 0 \text{ and } f_{xx} = -6 < 0 \to \text{LOCAL MAX}$$

(b) Bottom left is correct (max/min values have no level curves within it, and saddle points have cross-over shapes)

5. Consider a function $f(t) = f(x(t), y(t), z(t)) = xyz - z^2$, where $x(t)$, $y(t)$, $z(t)$ are defined as followed:

$$x(t) = 2t^2 + 1 \qquad y(t) = 3 - \frac{1}{t} \qquad z(t) = 3$$

Find the following values:

(a) $f_z(3, 1, 2)$ (b) $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t=0)}$ (c) $\left.\frac{df}{dt}\right|_{(t=1)}$

(a)

$$\frac{\partial}{\partial z}(xyz - z^2)$$

$$xy - 2z$$

$$\boxed{f_z(3, 1, 2) = 3 - 4 = -1}$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}t}x = 4t$$

$$\boxed{\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t=0)} = 0}$$

(c)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = yz(4t) + xz\left(\frac{1}{t^2}\right) + (xy - 2z)(0)$$

When $t = 1$:
$x = 3$
$y = 2$
$z = 3$

Plug into the equation:

$$\boxed{\left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{(t=1)} = 24 + 9 + 0 = 33}$$

6. If $f(x, y, z) = xye^z$, find the gradient of f and the directional derivative at (2,5,0) in the direction of $\vec{v} = 2\hat{i} - \hat{j} + \hat{k}$.

$f_x = ye^z$ $\quad$ $f_y = xe^z$ $\quad$ $f_z = xye^z$

The gradient of f is $\boxed{\nabla f(x, y, z) = \langle ye^z, xe^z, xye^z\rangle}$

The unit vector of $\vec{v}$ is

$$\vec{u} = \langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle$$

At (2,5,0),

$$\nabla f(2, 5, 0) = \langle 5, 2, 10\rangle$$

The directional derivative at (2,5,0) in the direction of $\vec{v}$ is calculated to be

$$D_u f(2, 5, 0) = \nabla f(2, 5, 0) \cdot \vec{u} = \langle 5, 2, 10\rangle \cdot \langle \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\rangle$$

$$= \frac{18}{\sqrt{6}} = \boxed{3\sqrt{6}}$$

7. Find min/max of f(x,y,z) = $3x^2 + 8y^2 + z^2 - 2z$ defined on the domain $x^2 + 4y^2 + 2z \leq 8$ and z ≥ 0

(a) The domain is (select all that apply)

I) open

II) closed

III) bounded

IV) unbounded

(b) Where are the critical points inside the domain? Evaluate the function value on these points.

(c) What is the minimum and maximum on $x^2 + 4y^2 + 2z = 8$?

(d) What is the minimum and maximum on z = 0?

(e) What is the global minimum and maximum of the whole domain?

(a) The region includes *all* of its boundary points, therefore it's closed and bounded (II and III).

(b) The critical points inside the domain are where $\nabla f = 0$.

$$\nabla f = \langle 6x, 16y, 2z - 2 \rangle = \langle 0, 0, 0 \rangle$$

Which gives us

$$x = 0, \ y = 0, \ z = 1$$

$$\boxed{\text{Critical Points: } f(0, 0, 1) = -1}$$

(c) Since we're dealing with the boundary now, we must use Lagrange multipliers. Start by defining $g(x, y, z)$ as the boundary of the curve with $g(x, y, z) = 8$.

$$g(x, y, z) = x^2 + 4y^2 + 2z$$
$$\nabla f = \lambda \nabla g \rightarrow \langle 6x, 16y, 2z - 2 \rangle = \langle 2\lambda x, 8\lambda y, 2\lambda \rangle$$

Move ∇f and $\lambda \nabla g$ to the same side to solve for each variable.

$6x - 2\lambda x = 0$
$\lambda = 3$ or $x = 0$

$16y - 8\lambda y = 0$
$\lambda = 2$ or $y = 0$

$2z - 2 - 2\lambda = 0$
$\lambda = z - 1$

$$g(x, y, z) = x^2 + 4y^2 + 3z = 12$$

Solving these four equations for four unknowns, we have three solutions:

$\lambda = 3, \ x = 0, \ y = 0, \ z = 4$

$\lambda = 2, \ x = 0, \ y = \pm\frac{1}{\sqrt{2}}, \ z = 3$

$$\boxed{f(0, 0, 4) = 8 \qquad f(0, \pm\frac{1}{\sqrt{2}}, 3) = 7}$$

(d) We are using a new boundary now, so we have to define a new $g(x, y, z)$.

$$g(x, y, x) = z$$

$$\nabla f = \lambda \nabla g \rightarrow \langle 6x, 16y, 2z - 2\rangle = \langle 0, 0, \lambda\rangle$$

$$g(x, y, z) = 0$$

There is one solution $x = 0,\ y = 0,\ z = 0,\ \lambda = -2$

$$\boxed{f(0, 0, 0) = 0}$$

(e)

$$\boxed{f(0, 0, 1) = -1 \qquad f(0, 0, 4) = 8}$$

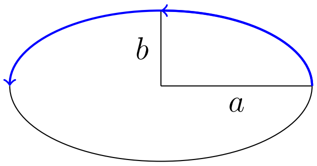
8. The vector field $\vec{F} = \langle 2xy + 2x + y^2, 2xy + 2y + x^2\rangle$ is conservative. Find a potential function f for $\vec{F}$ (a function with $\nabla f = \vec{F}$)

$$\vec{F} = \langle f_x, f_y\rangle = \langle 2xy + 2x + y^2, 2xy + 2y + x^2\rangle$$
$$\int f_x = yx^2 + x^2 + xy^2 + C(y)$$
$$\int f_y = xy^2 + y^2 + yx^2 + C(x)$$

Looking at all the terms and comparing with $\vec{F}$, we know that $C(y) = y^2$ and $C(x) = x^2$, therefore the potential function is:

$$\boxed{f(x, y, z) = yx^2 + x^2 + xy^2 + y^2}$$

9. A particle moves along the upper part of an ellipse in the xy-plane that has its center at the origin with semi-major and semi-minor axes $a = 4$ and $b = 3$, respectively. Starting at $(a, 0)$ and ending at $(-a, 0)$ and subject to the following force field, what is the total work done?



$$\vec{F} = (3x - 4y + 2z)\hat{\imath} + (4x + 2y - 3z^2)\hat{\jmath} + (2xz - 4y^2 + z^3)\hat{k}$$

(1) Find the parameterization of the ellipse

$$x = 4\cos t,\ y = 3\sin t,\ z = 0$$
$$dx = -4\sin t\ dt,\ dy = 3\cos t\ dt,\ dz = 0$$

Recall that $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

(2) Take the dot product in the line integral for work. (The z component is zero so it can be ignored here).

$$\oint \vec{F} \cdot \mathrm{d}\vec{r} = \int \left[(3x - 4y)\hat{\imath} + (4x + 2y)\hat{\jmath}\right] \cdot (dx\hat{\imath} + dy\hat{\jmath})$$
$$\oint (3x - 4y)dx + (4x + 2y)dy$$

(3) Substitute in the parameterization found in (1)

$$\oint (12\cos(t) - 12\sin(t))(-4\sin(t))dt + (16\cos(t) + 6\sin(t)(3\cos(t))dt$$

(4) Determine the times that the particle is at its starting and ending position, which in this case is $0 < t < \pi$. And solve the integral

$$\int_{t=0}^{t=\pi} \Big[(12\cos t - 12\sin t)(-4\sin t) + (16\cos t + 6\sin t)(3\cos t)\Big] dt$$

$$\boxed{48\pi}$$

10. Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f(x, y) = ye^{x^2-1} + 4x\sqrt{y}$ and C is given by $\vec{r}(t) = \langle 1 - t, 2t^2 - 2t\rangle$ with $0 \le t \le 2$.
Use the fundamental theorem of line integral.

$$\int_C \nabla f \cdot d\vec{r} = f[\vec{r}(2)] - f[\vec{r}(0)]$$

$$\vec{r}(0) = \langle 1, 0\rangle, \text{ and } \vec{r}(2) = \langle -1, 4\rangle$$

$$f[\vec{r}(2)] - f[\vec{r}(0)] = f(-1, 4) - f(1, 0) = -4 - 0 = \boxed{-4}$$