The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 2 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

DISCLAIMER: This review covers chapters taught in Sections A, B, and C. Section D covers different topics, but we can still answer questions after the presentation.

QR Code to the Queue

The queue contains the worksheet and the solution to this review session

Velocity and Acceleration

$$\boxed{\vec{r}'(t) = \vec{v}(t)} \longrightarrow \vec{r}(t) = \int \vec{v}(t)\,dt$$

$$\boxed{\vec{v}'(t) = \vec{a}(t)} \longrightarrow \vec{v}(t) = \int \vec{a}(t)\,dt$$

Gradient and Directional Derivatives

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The gradient will always point **perpendicular to the level curves/surfaces of f**
- ∇f = 0 at a local minimum/maximum

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

- Tells you how the function f changes along the vector u

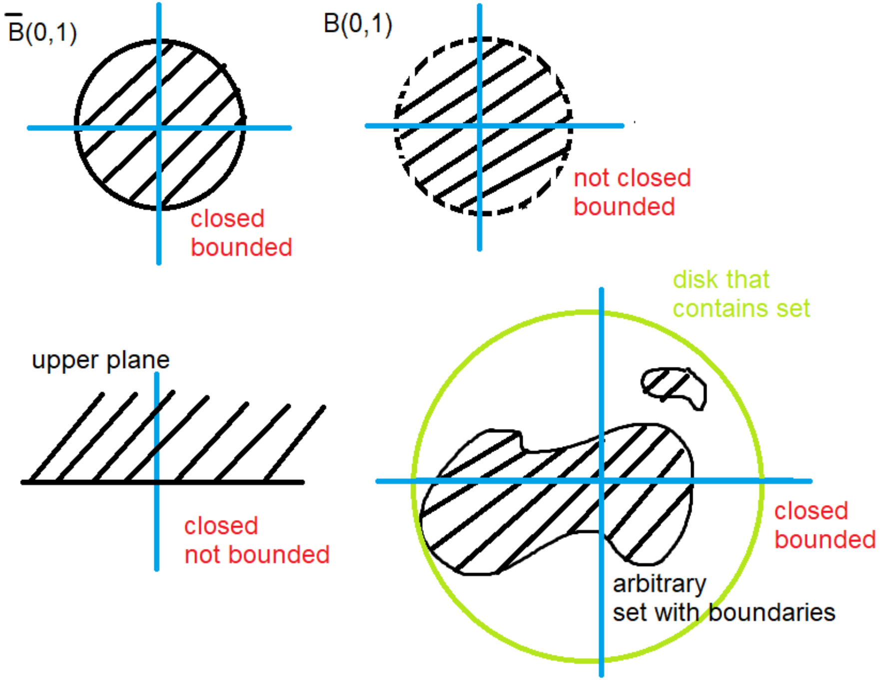
Arc Length Formula

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Extreme Value Theorem

If f(x,y) is continuous on a closed and bounded set D, then it is guaranteed that f has an absolute minimum and maximum value

- The absolute min and max will either occur at the critical points of f, or on the endpoints of the boundary D



https://math.stackexchange.com/questions/1190640/what-is-the-difference-between-closed-and-bounded-in-terms-of-domains

Local Min. & Max.

First Derivative Test:

For a continuous function, find the stationary point by solving gradient = 0.

The point (a, b) is a **critical point** (or a **stationary point**) of $f(x, y)$ provided one of the following is true,

1. $\nabla f(a, b) = \vec{0}$ (this is equivalent to saying that $f_x(a, b) = 0$ and $f_y(a, b) = 0$),
2. $f_x(a, b)$ and/or $f_y(a, b)$ doesn't exist.

Second Derivative Test

Distinguish between min. vs. max. at the point where gradient is 0.

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is: local min

If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is: local max

If $D < 0$, then $f(a,b)$ is: not critical

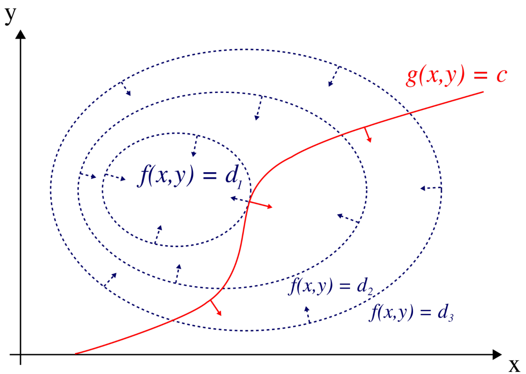
If $D < 0$, and $f_x = f_y = 0$,then $f(a,b)$ is: saddle

If $D = 0$, then $f(a,b)$ is: undetermined

Lagrange Multiplier

- Solve the following system of equations for λ (Lagrange Multiplier)
    - Where f is the function, and g is the constraint

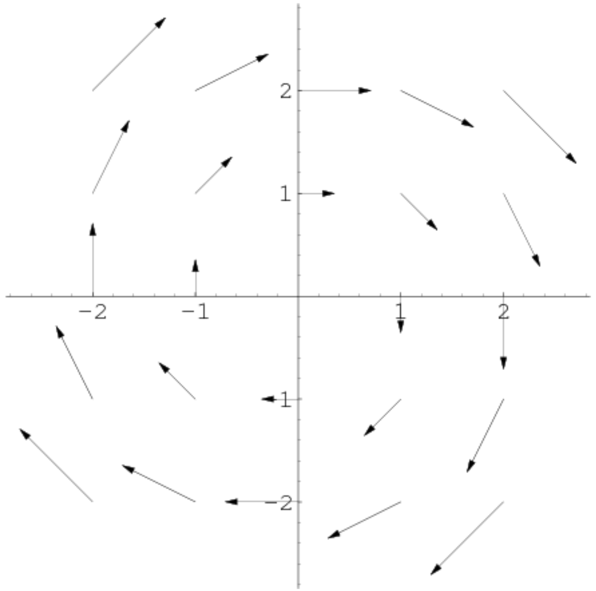
$$\nabla f\,(x,y,z) = \lambda\,\nabla g\,(x,y,z)$$
$$g\,(x,y,z) = k$$



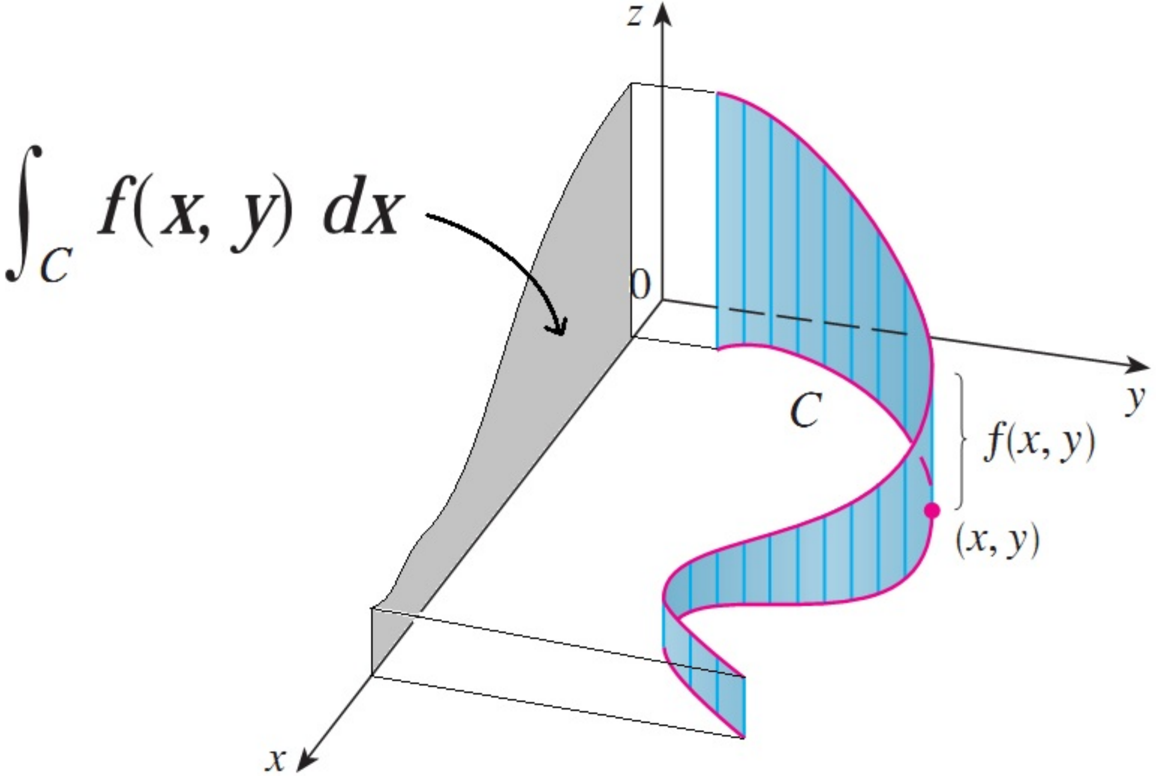
Vector Field

- A function that assigns a vector $\mathbf{F}$ to each point in 2D or 3D space.
- Takes in a point and “spits out” a vector

$$\vec{\mathbf{F}}(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}.$$



Intro Line Integrals



Line Integrals on Scalar Value Functions

- f takes in a vector and spits out a scalar
- ds is infinitesimal change on the curve C
- r(t) is a vector valued function which represents the curve C.

$$\int_C f(x,y)\,ds$$

$$\oint f(\mathrm{r})\mathrm{ds} = \int_a^b f[\mathrm{r}(t)]\,|\mathrm{r}'(t)|\,dt$$

$$\int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Line integrals of scalar function Part 2

$$\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$$

Line integral of f over a curve c with respect to a change in x,
<u>NOT</u> arc length

Line Integrals on Vector Value Functions

- F takes in a vector and spits out a vector
- r(t) is a vector function over time, which represents the curve C that we are integrating over

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$\downarrow$$

$$\int_C \langle g(x,y), h(x,y)\rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle\, dt$$

$$\downarrow$$

$$\int_C g\, dx + h\, dy$$

Fundamental theorem of line integrals

Suppose that C is a **smooth** curve given by $\vec{r}\,(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

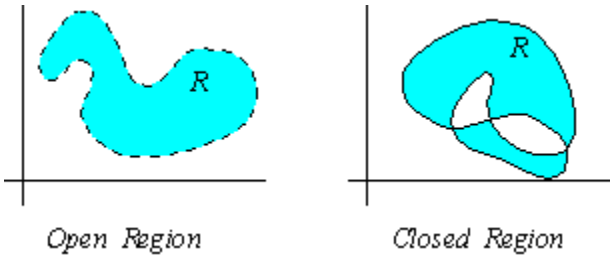
$$\int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}\,(b)\right) - f\left(\vec{r}\,(a)\right)$$

Compare to FTC:

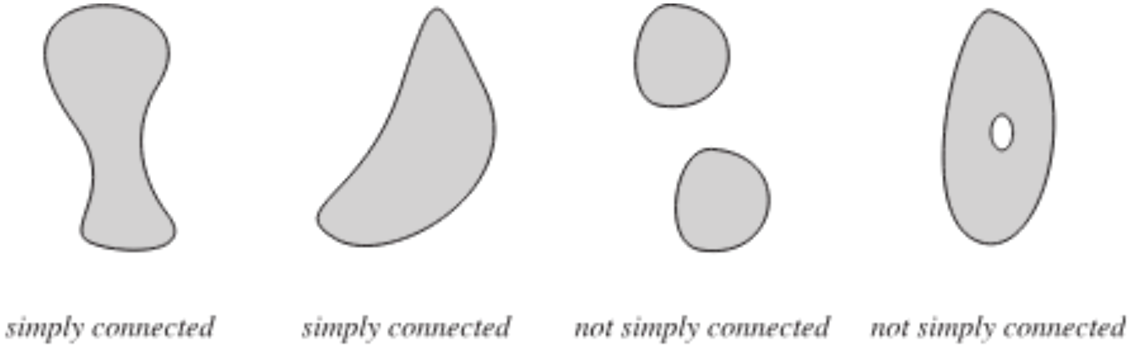
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Types of Regions

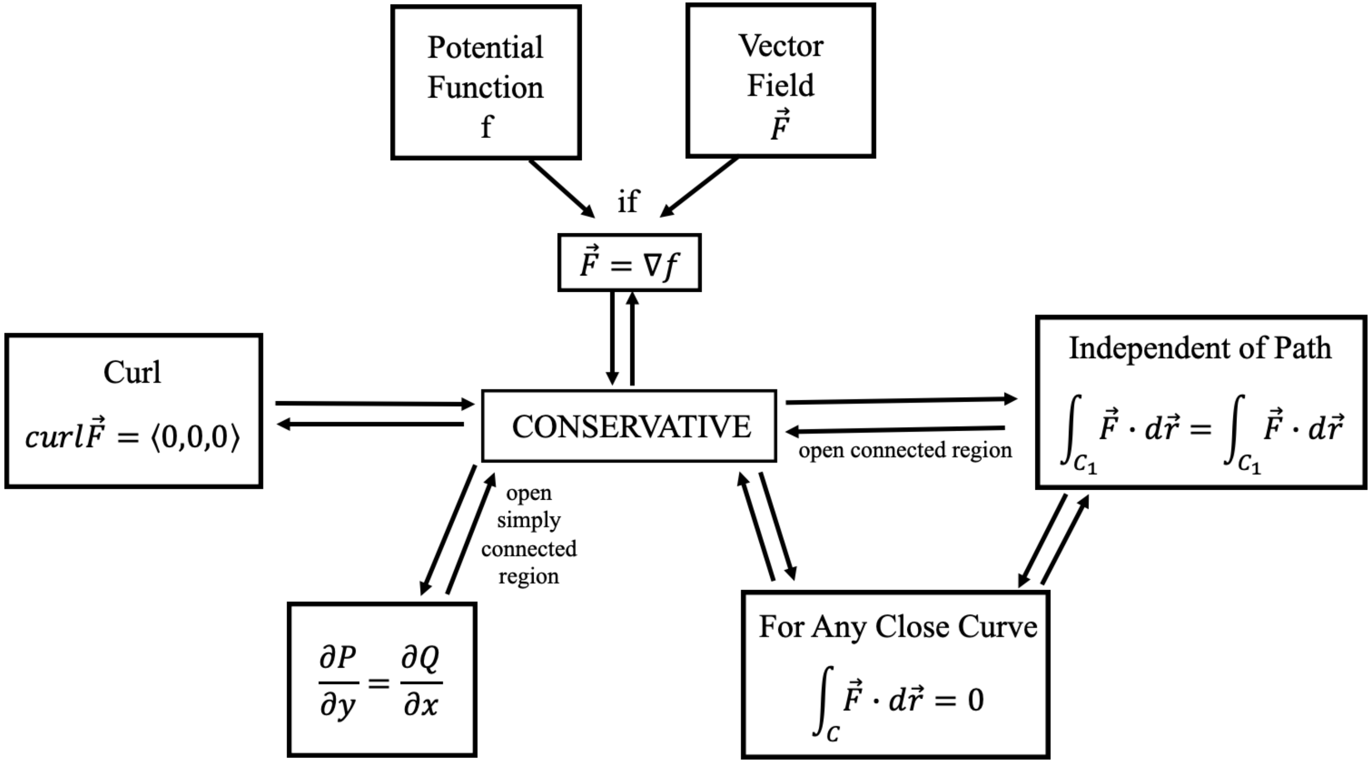
- Closed
  - DOES contain boundary
  - Open sets do not



- Connected:
  - Simply connected -> No holes
  - No partitions
  - DOES NOT have to be closed



Conservative Vector Field



Conservative Vector Field

- Line integrals of a conservative vector field are independent of path

$$\int_C F \cdot dr \text{ is independent of path D if and only if}$$

$$\int_C F \cdot dr = 0 \text{ for every closed path C in D}$$

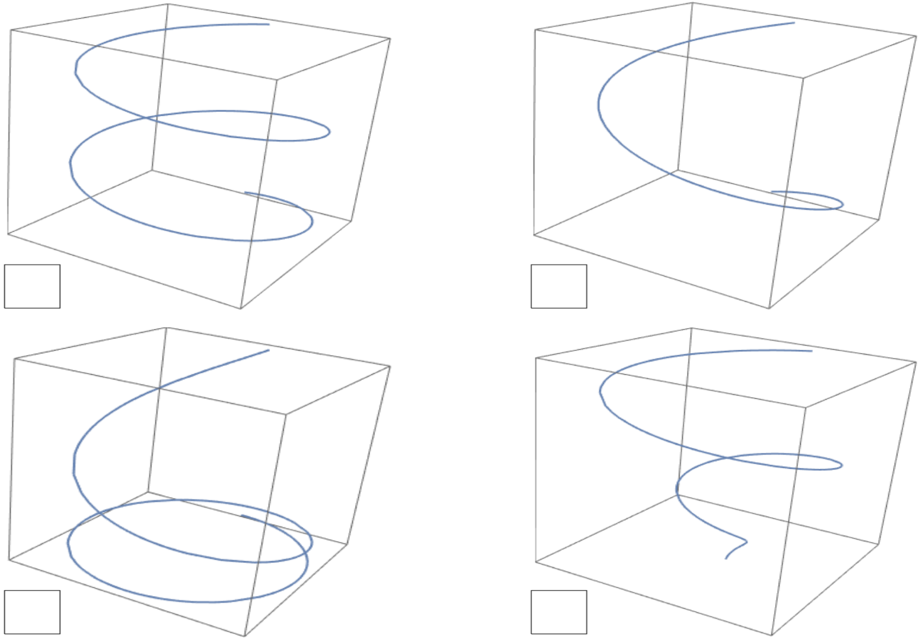
- Let F = P**i** + Q**j** be a vector field on an open simply-connected region D. Suppose that P and Q have continuous partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

, then F is conservative.

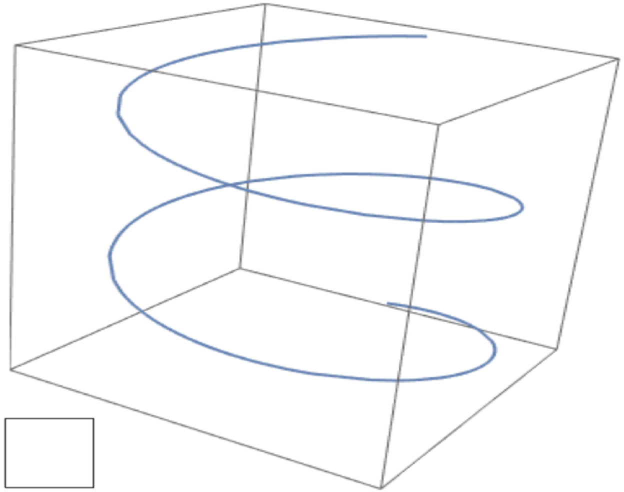
Example Question #1

Let C be the curve parameterized by $\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \le t \le 2\sqrt{\pi}$. Check the corresponding picture of C.



Example Solution #1

$\mathbf{r}(t) = \langle \sin(t^2), \cos(t^2), t^2 \rangle$ for $0 \le t \le 2\sqrt{\pi}$



Example Question #2

- Consider the function $f = x^3 + y^3 + 3xy$. If the critical points of f are $(0, 0)$ and $(-1, -1)$, classify them into local mins, maxes, and saddle points.

Example Solution #2

- Consider the function $f = x^3 + y^3 + 3xy$. If the critical points of f are (0, 0) and (–1, –1), classify them into local mins, maxes, and saddle points.

$$\mathrm{f_x} = 3\mathrm{x}^2 + 3y \quad \mathrm{f_y} = 3\mathrm{y}^2 + 3x \quad \mathrm{f_{xx}} = 6x \quad \mathrm{f_{yy}} = 6y$$

$$\mathrm{f_{xy}} = \mathrm{f_{yx}} = 3$$

$$\text{At } (0,0), \mathrm{D} = \begin{vmatrix} \mathrm{f_{xx}} & \mathrm{f_{xy}} \\ \mathrm{f_{yx}} & \mathrm{f_{yy}} \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 \rightarrow \text{Saddle Point}$$

$$\text{At } (-1,-1), \mathrm{D} = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 \rightarrow \text{Because } \mathrm{f_{xx}} = -6 < 0 \rightarrow \text{Local Max}$$