MATH 231 Exam Review

A Section Midterm 02

Content Covered During This Session

- 7.8: Improper integrals
- 8.1: Arc length
- 8.2: Area of a surface of revolution
- 8.3: Applications to physics and engineering
- 11.1: Sequences
- 11.2: Series

Improper Integrals:

There are two types:

1) Dealing with infinity

Example:

$$\int_0^{\infty} \frac{1}{x^4+1}\, dx$$

2) Dealing with a discontinuity

$$\int_{-1}^{1} \frac{1}{x}\, dx = 0$$

Convergent:

- Means that there is a finite answer

Divergent:

- Means that the integral does not exist or is infinite

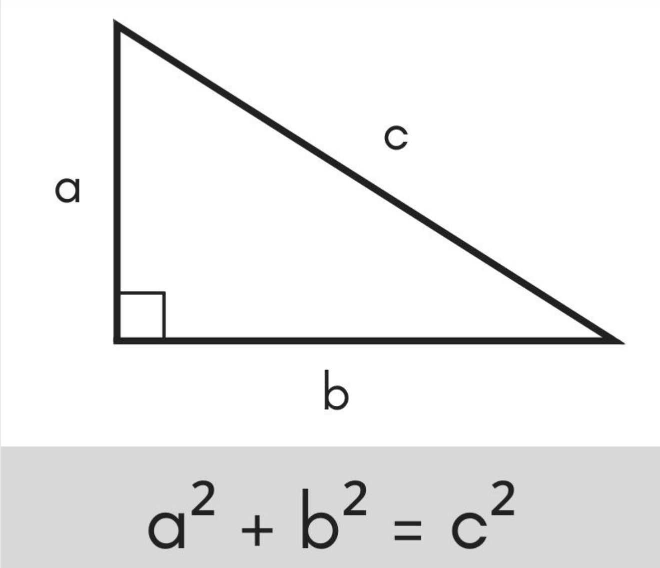
Setting up Improper Integrals

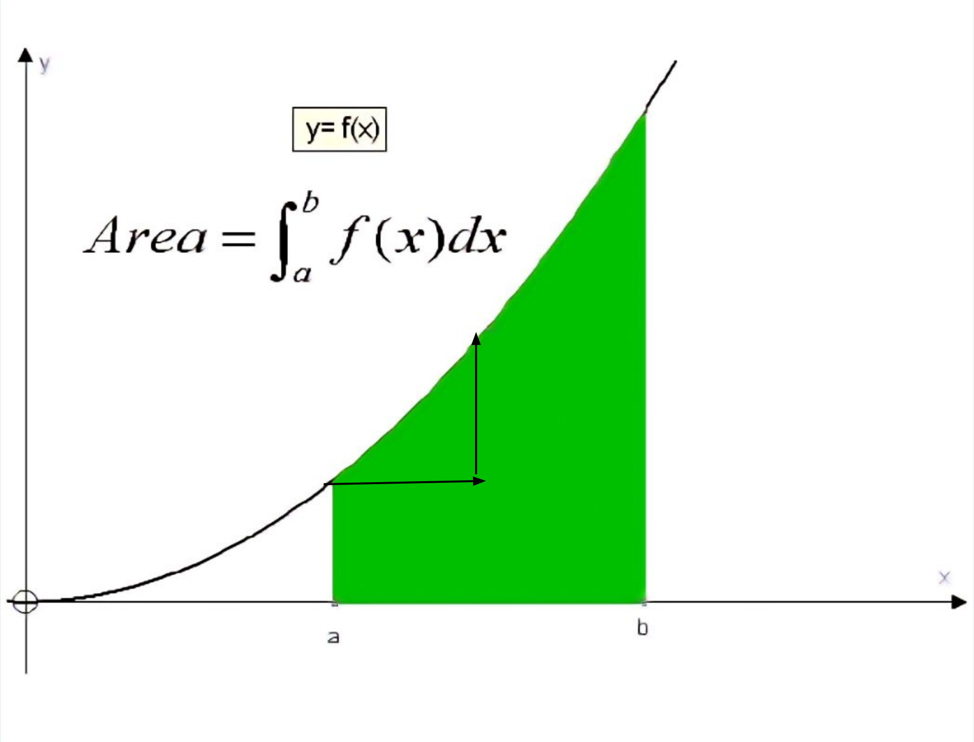
1. $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$ over the interval $[a,\infty)$

2. $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$ over the interval $(-\infty,b]$

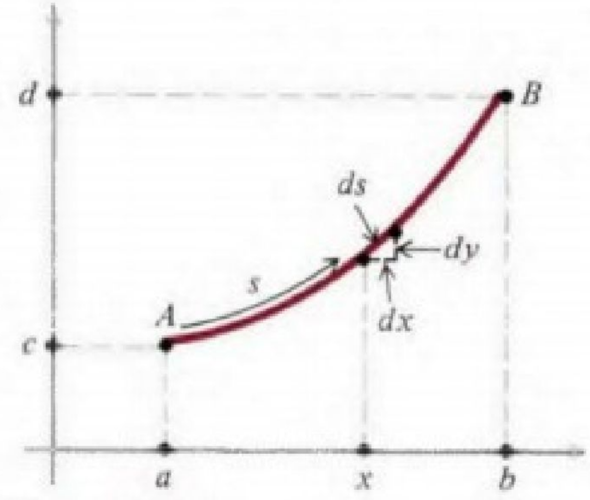
3. $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ over the interval $(-\infty,\infty)$

Arc Length: Simplest Case





Arc Length: Derivation



$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{\left(1 + \frac{dy^2}{dx^2}\right) dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

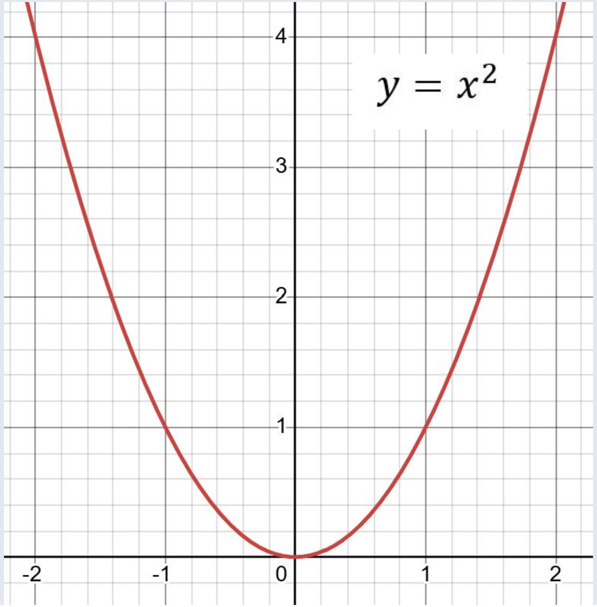
$$\boxed{\text{length of arc } AB = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

Two Formulas

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2 dx} \Rightarrow a \le x \le b \quad y = x^2$$

$$L = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2 dy} \Rightarrow a \le y \le b \quad x = \sqrt{y}$$

Which would you rather do?



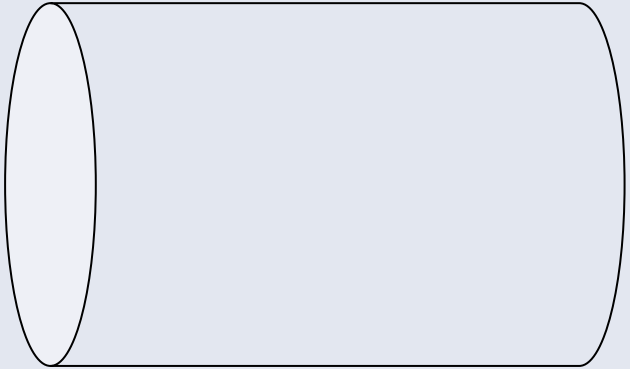
$$\boxed{\int_0^2 \sqrt{1+(2x)^2}\,dx =} \int_0^4 \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

General Steps

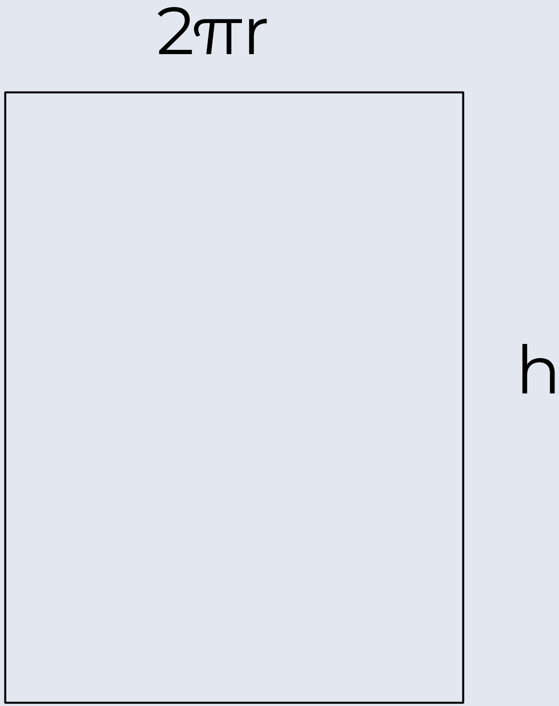
1. Write down formula that makes the most sense based on what you are given in the problem
2. Find the derivative
3. Set up the integrand and solve

$$y = x^2 \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2x \quad \Longrightarrow \quad \int_0^2 \sqrt{1 + (2x)^2}\,dx$$

Surface Area: Simplest Case

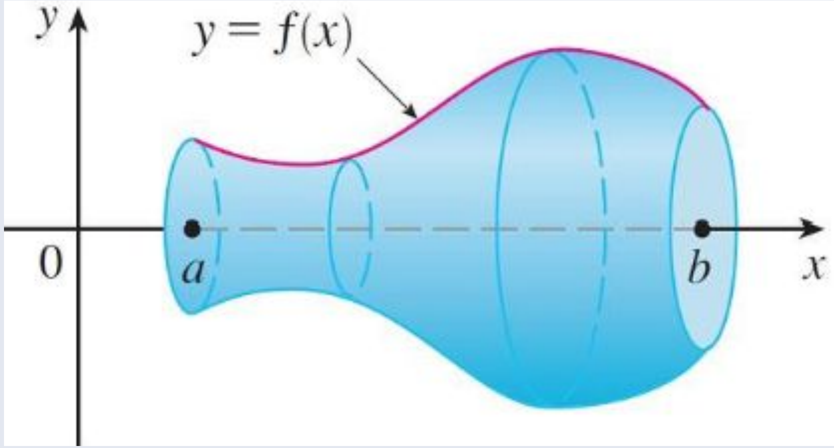


$A = 2\pi r \times h$



Surface Area of Revolution

We apply the same knowledge to more complex shapes.



The arc length will be the ‘h’, and then the given function will be your circumference.

Surface Area of Revolution: Formulas

The height of your function
[y = f(x)] is the radius.

$$S = \int 2\pi y \, ds$$

⟹ Around x-axis

$$S = \int 2\pi x \, ds$$

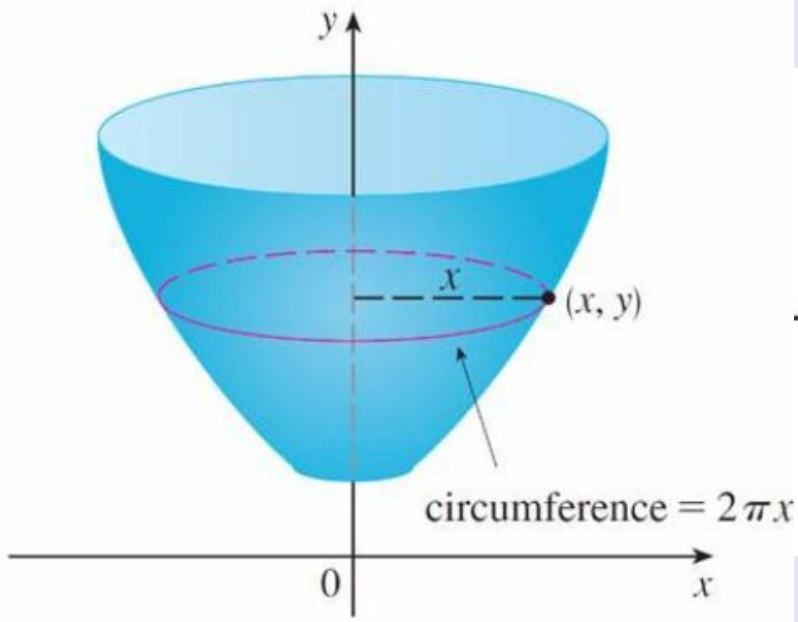
⟹ Around y-axis

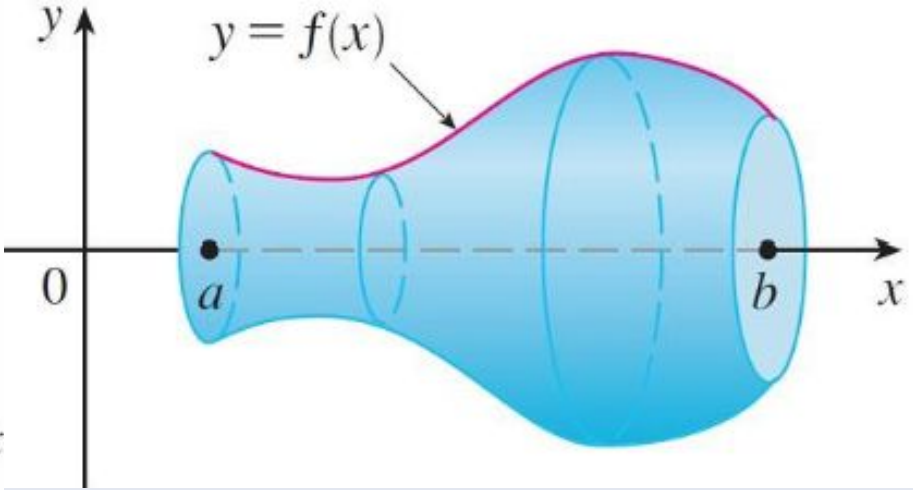
The width of your function
[x = f(y)] is the radius.

$$ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2 dx}$$

$$ds = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2 dy}$$

Visualizing what is going on



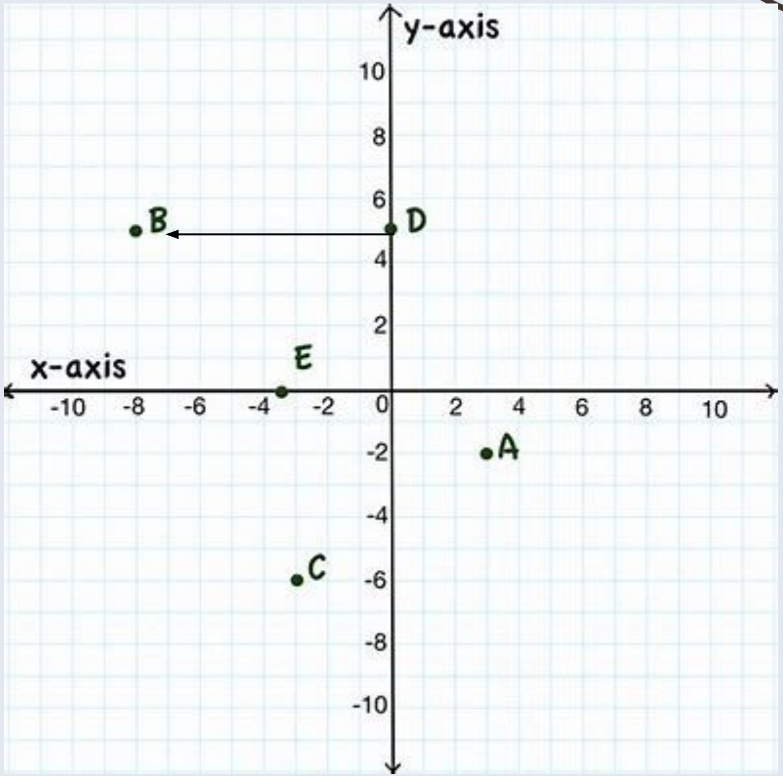


Moments and Center of Mass

Center of mass: $$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Moments about axis: $$M_y = \sum m_i x_i$$ $$M_x = \sum m_i y_i$$

Center of mass coordinates: $$\bar{x} = \frac{M_y}{m}$$ $$\bar{y} = \frac{M_x}{m}$$



Center of Mass with Uniform Density

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\,dx$$

If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, the centroid of $\mathcal{R}$ is $(\bar{x}, \bar{y})$, where

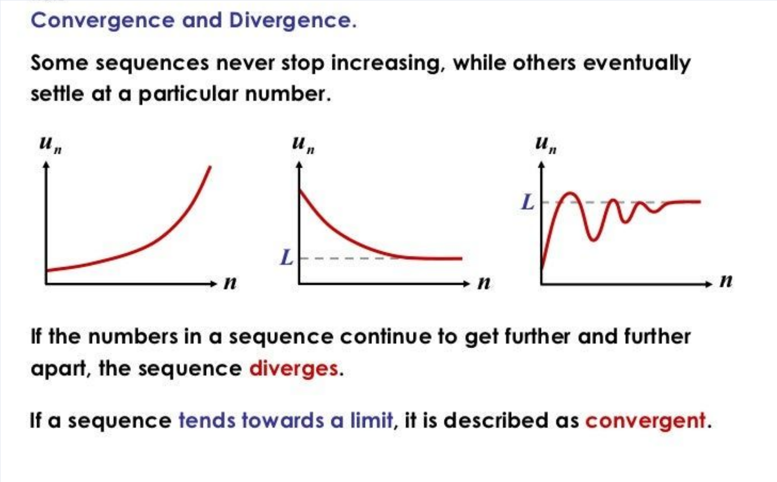
$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx$$

Sequences

Sequence: Just a list of the numbers

- Pattern Recognition!

- To test whether convergent or divergent take the limit, if the limit exists: convergent, if not! It is divergent.
  - If the sequence is a function, take the limit of the function
  - If cannot take limit, Squeeze Theorem!



Useful Squeeze Theorem

- $\lim\limits_{x \to \infty} \left(\frac{\sin(x)}{x}\right) = 0$
- $\lim\limits_{x \to 0} \left(\frac{\sin(ax)}{x}\right) = a$
- $\lim\limits_{x \to 0} \left(\frac{\sin(x)}{x}\right) = 1$
- $\lim\limits_{x \to 0} \left(\frac{\cos(x)-1}{x}\right) = 0$

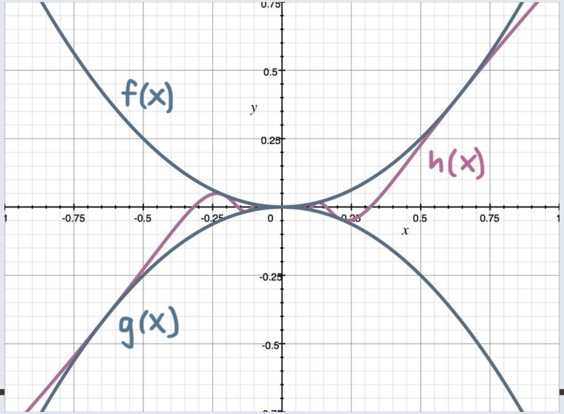
If $f(x) \leq h(x) \leq g(x)$

and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

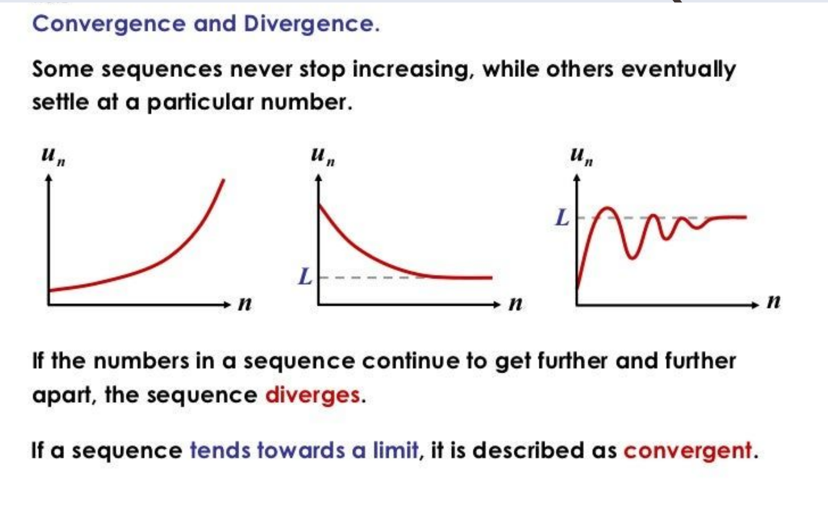
then

$$\lim_{x \to c} h(x) = L$$



Sequence Convergence

- Convergence:
  - Increasing
    - if all $a_n < a_{n+1}$
  - Decreasing
    - if all $a_n > a_{n+1}$
  - Bounded from Below
    - If there existed a number m such that $\mathrm{m} \leq a_{n+1}$
  - Bounded from Above
    - If there existed a number M such that $\mathrm{M} \geq a_{n+1}$
  - If a sequence is bounded (from above and below) and monotonic (increasing or decreasing only), then it is convergent
    - If is not both of these, does not necessarily mean it is divergent



Series

- Series: The sum of a sequence.
  - If a series converges, then the sequence must converge as well.
  - **However:** If sequence converges, then the series may or may not converge.
  - Σa_n converges if the limit of the series converges.
- Geometric series:
  - $\Sigma ar^{k-1} = a + ar + ar^2 + ar^3 \ldots.$
  - Will converge if $|r| < 1$
- Other techniques:
  - Evaluate the partial sums (first bit of sums) of a series and see how the series behaves
- If Σa_k converges, then $\lim_{x\to\infty} a_n = 0$

Questions?