Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Math 221 – Calculus I

Midterm 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: October 7, 4:00-5:30 pm Lucy and Patrick

Session 2: October 8, 6:00-7:20 pm Jiya and Patrick

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/1056
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Answer the following questions:

(a) State the Mean Value Theorem, emphasizing its key conditions.

Mean Value Theorem: Let f be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) State Rolle's theorem, emphasizing its key conditions.

Rolle's Theorem:

Let f be a function that satisfies the following conditions:

i. f is continuous on the closed interval $[a, b]$,
ii. f is differentiable on the open interval (a, b),
iii. $f(a) = f(b)$.

Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = 0.$$

2. Suppose that A represents the number of grams of a radioactive substance at time t seconds. Given that $\frac{dA}{dt} = -0.125A$, how long does it take 12 grams of the substance to be reduced to 4 grams?

First recall that $\frac{d}{dx} = ky$ so $y = ce^{kx}$. So $\frac{dA}{dt} = -0.125A$ and $A = ce^{-0.125t}$

Plugging in $A = 12$ when $t = 0$ gives us $12 = ce^{-0.125*0}$ gives us $12 = c$. Thus, $A = 12e^{-0.125t}$

$$4 = 12e^{-0.125t}$$

$$\ln\left(\frac{4}{12}\right) = -0.125t$$

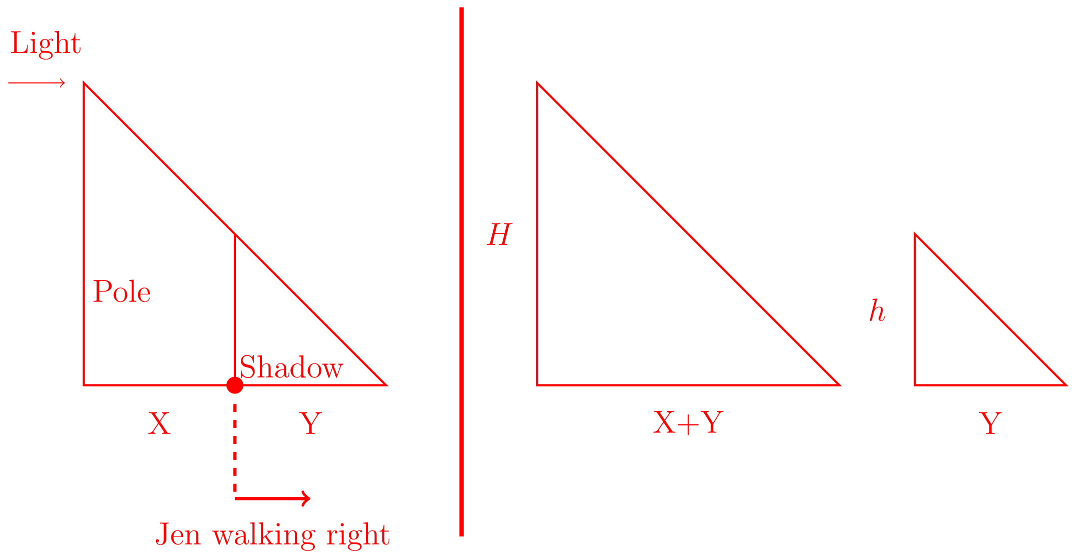
$$t = \frac{\ln\left(\frac{1}{3}\right)}{-0.125} = 8\ln(3)$$

$$t = \boxed{8\ln(3)\text{ s}}$$

3. A streetlight is mounted at the top of a tall pole with $H = 16.5$ ft. Jennifer's height is $h = 5.5$ ft tall. She walks away from the pole with a speed of 8 ft/s along a straight path. how quickly is the length of her shadow on the ground increasing when she is 15 ft from the pole?

Given: $\frac{dx}{dt} = 8$, we want $\frac{dy}{dx}|_{x=15}$

Use the below diagrams to help solve the problem



$$\frac{X+Y}{H} = \frac{Y}{h} = X + Y = \frac{Y}{h} * H = 3Y$$

$$X = 2Y$$

$$\frac{d}{dt}(X) = \frac{d}{dt}(2Y)$$

$$\frac{dx}{dt} = 2\frac{dy}{dt}$$

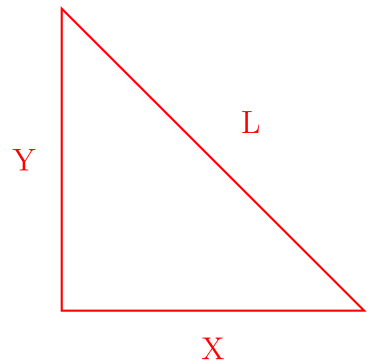
$$8 = 2\frac{dy}{dt}$$

$$\frac{dy}{dt} = 4$$

Therefore, the shadow length is increasing at a rate of $\boxed{4 \text{ ft/s}}$.

4. The top of a ladder slides down a vertical wall at a rate of 8 m/s. At the moment when the bottom of the ladder is 4 meters from the wall, it slides away from the wall at a rate of 15 m/s. How long is the ladder?

Note that L is a constant length.



Given: $\frac{dy}{dt} = -8$ and $\frac{dx}{dt}|_{x=4} = 15$

We want L:

$$X^2 + Y^2 = L^2$$

$$\frac{d}{dt}(X^2 + Y^2) = \frac{d}{dt}(L^2)$$

$$2X * \frac{dx}{dt} + 2Y * \frac{dy}{dt} = 0$$

$$2 * 4 * 15 + 2 * Y * (-8) = 0$$

$$Y = \frac{15}{2}$$

$$L = \sqrt{X^2 + Y^2}$$

$$\boxed{L = 8.5 \text{ m}}$$

5. Find the absolute minimum y-value of the given function:

$$y = \frac{2x}{\sqrt{x-81}}$$

Domain of the function: $x > 81$.

$$y' = \frac{2 * \sqrt{x-81} - (2x) * (\frac{1}{2}) * (x-81)^{-\frac{1}{2}}}{(\sqrt{x-81})^2}$$

$$y' = \frac{2\sqrt{x-81} - \frac{x}{\sqrt{x-81}}}{x-81}$$

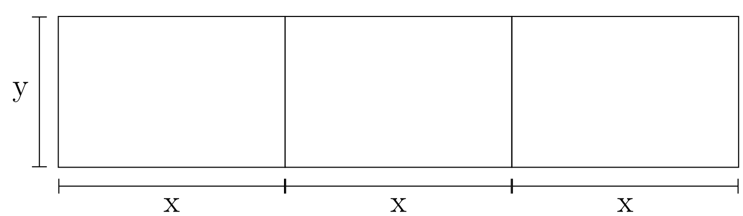
$$y' = \frac{2(x-81) - x}{(x-81)(\sqrt{x-81})}$$

$$y' = \frac{x-162}{(x-81)\sqrt{x-81}}$$

Setting $y' = 0$ allows us to find a relative minimum at $x = 162$. The minimum is $\boxed{36}$, which can be found by plugging 162 into the function for y.

However, ensure to check limits to see if this is an absolute minimum since the function hasn't been bounded. For this function, as x approaches infinity, the limit approaches infinity. Limiting to $x = 81$ from the right, the function also approaches infinity. Therefore $y(162) = 36$ is the absolute minimum. Ensure to review the Extreme Value Theorem for more information.

6. A farmer wishes to fence off three identical adjoining rectangular pens as in the diagram shown, but only has 600 feet of fencing available. Determine the values for x and y which will maximize the total area enclosed by these three pens.



Area $= 3xy$
Total fencing $= 600 = 6x + 4y$

$$4y = 600 - 6x$$

$$y = 150 - 1.5x$$

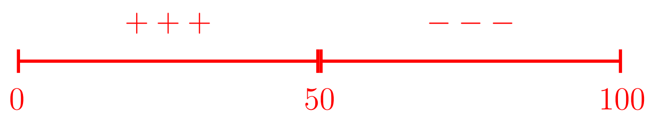
Area is now equivalent to:

$$\text{Area} = 3x * (150 - 1.5x)$$
$$\text{Area} = 450x - 4.5x^2$$

Now, we must maximize A for x in the range of $(0, 100)$

$$0 = \frac{dA}{dx}$$
$$0 = 450 - 9x$$
$$9x = 450$$
$$x = 50$$

Check for the values of A':



So we can see that there is an absolute maximum at $x = 50$. Evaluate y at $x = 50$

$$y = 150 - 1.5 * (50)$$
$$y = 75$$

$$\boxed{Area = 3 * 50 * 75 = 11,250 \text{ ft}^2}$$

7. Without using any kind of computational aid, use a linear approximation to estimate the value of $e^{0.1}$.

To estimate $e^{0.1}$ using linear approximation, we first recognize that the exponential function $f(x) = e^x$ is smooth and continuous, making it a good candidate for linearization near $x = 0$.

$$f(x) \approx f(0) + f'(0)(x - 0).$$

For $f(x) = e^x$:

$$f(0) = e^0 = 1 \quad \text{and} \quad f'(x) = e^x.$$

Thus:

$$f'(0) = e^0 = 1.$$

Now, applying the linear approximation formula at $x = 0.1$:

$$e^{0.1} \approx 1 + (1)(0.1) = 1 + 0.1 = 1.1.$$

Therefore, using linear approximation, we estimate that:

$$e^{0.1} \approx 1.1.$$

8. Given the function $f(x) = 5x^2 - 3x + 15$

(a) Decide whether the function is increasing or decreasing at $x = 0$ and $x = 1$.

(b) Find the critical points of the function and state whether they are maximums or minimums using the first derivative test.

(c) Use the second derivative test to check whether the critical points are maximums or minimums. Does your answer agree with part (b)?

(a) First, we find the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}(5x^2 - 3x + 15) = 10x - 3.$$

At $x = 0$:

$$f'(0) = 10(0) - 3 = -3.$$

Since $f'(0) < 0$, the function is decreasing at $x = 0$.
At $x = 1$:

$$f'(1) = 10(1) - 3 = 7.$$

Since $f'(1) > 0$, the function is increasing at $x = 1$.

(b) Critical points occur when $f'(x) = 0$. Setting $f'(x) = 0$:

$$10x - 3 = 0 \quad \Rightarrow \quad x = \frac{3}{10}.$$

To determine whether this critical point is a maximum or minimum, we apply the first derivative test. We evaluate $f'(x)$ on intervals around $x = \frac{3}{10}$.
For $x < \frac{3}{10}$ (e.g., at $x = 0$):

$$f'(0) = -3 \quad \text{(negative)},$$

so f is decreasing.
For $x > \frac{3}{10}$ (e.g., at $x = 1$):

$$f'(1) = 7 \quad \text{(positive)},$$

so f is increasing.
Since $f'(x)$ changes from negative to positive at $x = \frac{3}{10}$, f has a local minimum at $x = \frac{3}{10}$.

(c) We calculate the second derivative of $f(x)$:

$$f''(x) = \frac{d}{dx}(10x - 3) = 10.$$

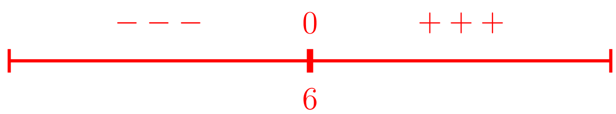
Since $f''(x) = 10 > 0$ for all x, the function is concave up, meaning that the critical point $x = \frac{3}{10}$ is a local minimum. The second derivative test confirms that the critical point at $x = \frac{3}{10}$ is a local minimum, which agrees with the result from the first derivative test.

9. A function $f(x)$ has the first derivative $f'(x) = e^{0.5x}(10x - 60)$

(a) Upon which interval is $f(x)$ increasing?

(b) Upon which interval is the graph of $f(x)$ concave down?

(a) Based off of the graph below, we can say f is increasing on the interval $(6, \infty)$. So the answer is $\boxed{(6, \infty)}$.



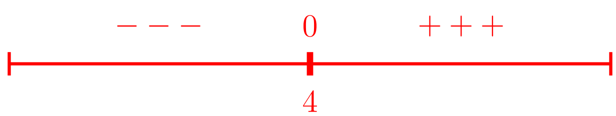
(b)

$$f''(x) = 0.5e^{0.5x} * (10x - 60) + e^{0.5x} * 10$$

$$f''(x) = e^{0.5x}(0.5(10x - 60) + 10)$$

$$f'(x) = e^{0.5x}(5x - 20)$$

Values of f”(x):



The function is concave down on the interval $\boxed{(-\infty, 4)}$

10. Evaluate each of the following limits:

(a)

$$\lim_{x \to \infty} \frac{2\ln(x)}{\sqrt[3]{x}}$$

(b)

$$\lim_{x \to 0} \frac{e^{10x} - 1}{5x}$$

(c)

$$\lim_{x \to \infty} \frac{e^{10x} - 1}{5x}$$

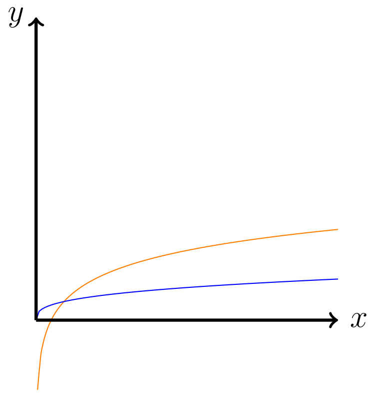
(a)

$$\lim_{x\to\infty} \frac{2\ln(x)}{\sqrt[3]{x}} = 2 \lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}}$$

Using l'Hopital's Rule:

$$2 \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-2/3}} = 2 \lim_{x\to\infty} \frac{3}{x^{1/3}} = 2(0) = \boxed{0}$$

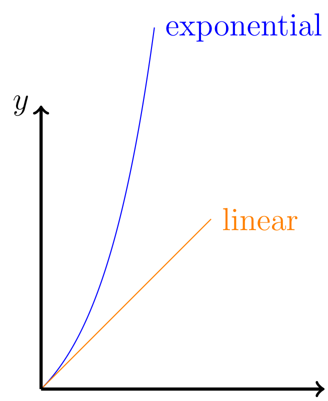
$$\left(\text{Blue} = \sqrt[3]{x}, \text{ orange} = 2\ln(x)\right)$$



(b) For this limit we must apply l'Hopital's Rule

$$\lim_{x\to 0} \frac{e^{10x}-1}{5x} = \lim_{x\to 0} \frac{10e^{10x}}{5} = \boxed{2}$$

(c) The numerator approaches infinity more rapidly than the denominator $\boxed{\lim_{x\to\infty} \to \infty}$



(Note about plots, these plots do not have the coefficients attached to x due to size constraints of the page. The general relation still holds and, in fact, is exacerbated by the coefficients)

11. Evaluate the derivative of the following:

(a)

$$f(x) = sin(x)^{ln(x)}$$

(b)

$$g(x) = \frac{1}{cos^{-1}(x^3 + x)}$$

(a)

$$f(x) = sin(x)^{ln(x)}$$

$$f(x) = y$$

Rewrite using logarithmic properties:

$$ln(a^b) = b(ln(a))$$

$$ln(y) = ln(sin(x)^{ln(x)})$$
$$ln(y) = ln(x)ln(sin(x))$$

Implicit Derivation and Product Rule

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}ln(x)(ln(sin(x)) + ln(x)\frac{d}{dx}(ln(sin(x))$$

$$\frac{d}{dx}ln(x) = \frac{1}{x}$$
$$\frac{d}{dx}ln(sin(x)) = \frac{1}{sin(x)}cos(x) = cot(x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{ln(sin(x))}{x} + ln(x)cot(x)$$
$$\frac{dy}{dx} = y(\frac{1}{x}ln(sin(x)) + ln(x)cot(x))$$
$$\frac{dy}{dx} = sin(x)^{ln(x)}(\frac{1}{x}ln(sin(x)) + ln(x)cot(x))$$

(b)

$$g(x) = \frac{1}{cos^{-1}(x^3 + x)}$$

Define: $h(x) = cos^{-1}(x^3 + x)$

$$g(x) = (h(x))^{-1}$$

$$g'(x) = -(h(x))^{-1}h'(x)$$

$$\frac{d}{dx}[cos^{-1}(u)] = \frac{-u'}{1 - u^2}$$

$$h'(x) = \frac{-(3x^2 + 1)}{\sqrt{1 - (x^3 + x)^2}}$$

$$g'(x) = \frac{-(3x^2 + 1)}{\sqrt{1 - (x^3 + x)^2}}(-(cos^{-1}(x^3 + x))^{-2})$$

$$g'(x) = \frac{3x^2 + 1}{\sqrt{1 - (x^3 + x)^2}(cos^{-1}(x^3 + x))^2}$$