# Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

**PHYS 214 – University Physics: Quantum Physics**

**Midterm 2 Worksheet Solutions**

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*The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.*

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Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: 2/18, 6-8 pm in 4035 CIF, led by Aparna, Sarah, and Zaahi

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

**Please be sure to follow the above steps to add yourself to the Queue.**

Good luck with your exam!

1. Consider the wavefunction $\Psi(\mathrm{x}) = \mathrm{N}e^{ikx}$, $0 \leq x \leq L$

a) What is the momentum of a particle given by $\Psi$(x)?

b) Find the probability density and show that it is constant in position over the given interval.

c) Knowing that this wavefunction has definite momentum (momentum eigenstate), what can be said of its position?

(a)

$$p = \hbar k$$

Conceptually, given a particle with this wavefunction, any measurement of momentum will result in the above product.

(b)

$$\rho(x) = \Psi\Psi^* = (Ne^{ikx})(Ne^{-ikx}) = N^2$$

and the normalization constant does not depend on position here.

(c) By the Uncertainty Principle, definite momentum means indefinite position. This particle can be anywhere where the wavefunction is not equal to zero, meaning we know very little about where the particle is located.

2. In a photoelectric effect demonstration, the intensity of the incident light is gradually increased, but no photocurrent is detected. Provide an explanation for this result.
It's the energy of a photon, not intensity, that increases the kinetic energy of the photoelectron (more specifically, the *frequency* of a photon). Since no current is detected, the energy of the incoming photons is less than the work function of the material, so no electrons escape. The frequency must be increased to change this.

3. A laser with time-varying frequency is directed at a barrier with a narrow slit followed by a screen. Assuming the laser intensity is constant, as the frequency increases, how does the number of photons per second arriving at the screen change?
The number of photons arriving at the screen will decrease. Using dimensional analysis:

$$\frac{\text{photons}}{\text{second}} * \frac{\text{joules}}{\text{photon}} = \frac{\text{joules}}{\text{second}}$$

The right hand side is proportional to the intensity (W/m$^2$), which is held constant. So if the frequency increases, the energy per photon increases, and the left hand side increases (J/photon). To keep intensity constant, the number of photons per second must decrease

4. A spacecraft is being pushed by a laser of wavelength 400 nm emitting photons at a rate of $10^{22}$ photons per second. Calculate the acceleration of the spacecraft given its mass is 4000 kg. Values are given in meters per second.

a) $3.25 \times 10^{-6}$

b) $1.53 \times 10^{-3}$

c) $4.14 \times 10^{-9}$

d) $1.7 \times 10^{-5}$

e) $5.5 \times 10^{-4}$

The answer is **(c)**. The photon's momentum can be found by

$$p = \frac{h}{\lambda}$$

which can be multiplied by the rate at which photons are emitted, $R_\gamma$ to give us the recoiling force by Newton's second law:

$$F = \frac{dp}{dt} = pR_\gamma \equiv \frac{\text{momentum}}{\text{photon}} * \frac{\text{photons}}{\text{second}} = \frac{\text{momentum}}{\text{sceond}} = \text{Newtons}$$

This force, divided by the ship's mass giving us the acceleration

5. Verify the following identity: $\sin(kx) = \frac{(e^{ikx} - e^{-ikx})}{2i}$ . Hint:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Euler's Identity

$$e^{ix} = \cos(x) + i\sin(x)$$

Therefore

$$e^{ikx} = \cos(kx) + i\sin(kx) \textbf{ and } e^{-ikx} = \cos(kx) - i\sin(kx)$$

Subtracting these we get $2i\sin(kx)$ , then dividing by $2i$ we get our final result.

6. Light with wavelength 100 nm is incident on a metal. The speed of the ejected photoelectrons is measured to be $10^6$ meters per second. Find the work function of this metal.

a) $1.99{\times}10^{-18}$

b) $1.53{\times}10^{-18}$

c) $4.55{\times}10^{-18}$

The answer is **(b)**. Through conservation of energy, we know that

$$hf = \frac{1}{2}mv^2 + \Phi$$

The frequency of this light can be found by dividing the speed of light by its wavelength

$$f = \frac{3 * 10^8}{100 * 10^{-9}} = 3 \times 10^{15} \text{ Hz}$$

Multiplying this with Planck's constant h gives us the total energy of the incoming photons:

$$1.988 \times 10^{-18} \text{ Joules}$$

The kinetic energy of the ejected electrons is

$$\frac{1}{2}mv^2 = 4.55 \times 10^{-19} \text{ Joules}$$

The difference in these values given the work function of this metal, using the above conservation equation.

$$\boxed{1.988 \times 10^{-18} - 4.55 \times 10^{-19} = 1.53 \times 10^{-18} \text{ J}}$$

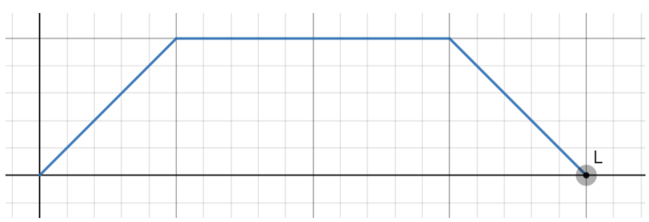
7. A particle's wave function is given by

$$\Psi(x) = \begin{cases} Ax & \text{if } 0 \leq x < L/4 \\ AL/4 & \text{if } L/4 < x < 3L/4 \\ -A(x-L) & 3L/4 \leq x \leq L \\ 0 & \text{elsewhere} \end{cases}$$

a) Sketch this wave function.

b) Without integrating, find the probability of finding the particle between $x = 0$ and $x = L/2$.

c) Find the normalization constant $A$ in terms of $L$.

a) 


b) $P = 0.5$. This is due to symmetry.

c) $A = 2\sqrt{\frac{6}{L^3}}$. We get the probability density by multiplying the wave function by its complex conjugate (which in this case is itself). We then integrate the probability density over 0 to $L$, knowing that it should equal 1, and solve for $A$:

$$\begin{aligned} 1 &= \int_0^{L/4} A^2x^2 \,\mathrm{d}x + \int_{L/4}^{3L/4} A^2\frac{L^2}{16} \,\mathrm{d}x + \int_{3L/4}^{L} A^2(x-L)^2 \,\mathrm{d}x \\ &= A^2\frac{L^3}{24} \\ A &= 2\sqrt{\frac{6}{L^3}} \end{aligned}$$

8. A material with work function $\Phi = 3.4$ eV has a laser beam with $\lambda = 200$ nm and power $P = 2.3 \times 10^{-4}$ W.

a) Calculate $N_\gamma$, the number of photons hitting the material per second.

b) Calculate the energy $E_{e^-}$, the maximum energy of each ejected electron.

c) Say we have a device that detects the power of the ejected electrons. Calculate the maximum power $P$ this device could measure (assuming every photon ejects an electron, and each electron has maximum energy).

a) $\boldsymbol{N_\gamma = 2.319 \times 10^{14}}$ photons/s. We know the power, and the energy of each photon is equal to $hc/\lambda = (1240\text{ eV}\cdot\text{nm})/(200\text{ nm}) = 6.2\text{ eV} = 9.92 \times 10^{-19}$ J. Performing units analysis:

$$N_\gamma = \frac{2.3 \times 10^{-4}\text{ J}}{\text{s}} \cdot \frac{\text{photon}}{9.92 \times 10^{-19}\text{ J}} = 2.319 \times 10^{14}\text{ photons/s.}$$

b) $\boldsymbol{E_{e^-} = 2.8}$ eV. The energy of each photon is equal to 6.2 eV. Subtracting the work function gives $E_{e^-} = 2.8$ eV.

c) $\boldsymbol{1.039 \times 10^{-4}}$ W. If each photon is ejecting an electron of maximum energy, then $2.319 \times 10^{14}$ electrons/s are being ejected, each with energy 2.8 eV. Multiplying these values yields $P = 6.492 \times 10^{14}\text{ eV/s} = 1.039 \times 10^{-4}$ W.

9. Confirm that the wave function

$$\Psi(x) = \frac{1}{5}\left(e^{i\pi/2} + 4i\right)$$

is normalized over the range $0 < x < 1$.

The probability density is given by

$$\begin{aligned}|\Psi(x)|^2 &= \Psi^*(x)\Psi(x)\\ &= \frac{1}{25}\left(e^{-i\pi/2} - 4i\right)\left(e^{i\pi/2} + 4i\right)\\ &= \frac{1}{25}\left(1 - 4ie^{i\pi/2} + 4ie^{-i\pi/2} + 16\right)\\ &= \frac{1}{25}(1 + 4 + 4 + 16)\\ &= 25/25 = 1\end{aligned}$$

Here, we used Euler's identity $\left(e^{i\theta} = \cos\theta + i\sin\theta\right)$ to find that $e^{i\pi/2} = i$ and $e^{-i\pi/2} = -i$. Integrating 1 over $0 < x < 1$ yields 1, so this wavefunction is normalized.

10. We have an electron double slit experiment. Describe what will happen to the spacing of the fringes if we:

a) decrease the slit separation

b) decrease the screen distance

c) send fewer electrons per second

d) replace the electrons with neutrons of the same momentum

e) send the electrons with greater momentum

We can refer to the equation $d \sin\theta = m\lambda$ as well as the De Broglie relation $p = h/\lambda$ for our answers.

a) Fringes grow farther apart.

b) Fringes grow closer together.

c) Fringes are unchanged.

d) Fringes remain unchanged.

e) Fringes grow closer together.

11. A baseball pitcher throws a standard baseball (m = 0.145 kg) at a speed of 35 m/s. Determine the baseball's De Broglie wavelength, and use this to explain why we don't see interference patterns in day-to-day life (unless you're a physicist).
p = mv = $h/\lambda$, so $\lambda = h/mv = 1.306 \times 10^{-34}$ m. This wavelength is extremely small, meaning that any interference pattern of baseballs would have fringes that essentially blend into one another.

12. Suppose we have a simple wave function (a momentum eigenstate), $\Psi(x) = Ne^{ikx}$, where $N$ is the normalization constant and $k$ is the wave number.

(a) What is the certainty in momentum (how many values could we possibly measure)?

(b) What is the certainty in position (what does the probability density look like)?

(c) Describe how parts (a) and (b) are consistent with Heisenberg uncertainty.

(d) Now, suppose we add another momentum eigenstate such that the wave function now reads as $\Psi(x) = N(e^{ikx} + e^{-ikx})$. Repeat parts (a) - (c) for this new wave function.

(a) There is only one possible value for momentum $p = \hbar k$, so momentum is the most certain it could be.

(b) The probability density is $N^2$, which is constant, meaning that the particle could be anywhere with equal probability. Thus position is the least certain it could be.

(c) In this scenario, momentum is extremely certain while position is extremely uncertain. We know that according to Heisenberg uncertainty, the certainties of momentum and position should roughly be inversely proportional, so this situation makes sense (one is big while one is small).

(d) Adding another momentum eigenstate, there are now *two* possible values for momentum, $p = \pm\hbar k$, so momentum is more uncertain than before. As for position, using Euler identities yields $\Psi(x) = 2N\cos(kx)$ and $\rho(x) = 4N^2\cos^2 kx$. The probability density is no longer constant; it has a distinct peak. Thus position is more certain than before. Again, this is consistent with Heisenberg uncertainty: as momentum becomes less certain, position becomes more certain.

13. A particle's wave function is given by

$$\Psi(x) = A\Big(3\, e^{ik_1x} + (1+i)\, e^{-ik_1x} + 2\, e^{ik_2x}\Big)$$

a) List the possible momentum eigenvalues of the particle.

b) What is the probability of measuring the momentum

$$p = +\hbar k_1\,?$$

a) The momentum eigenvalues correspond to the exponential terms:

$$p = +\hbar k_1, \quad p = -\hbar k_1, \quad p = +\hbar k_2.$$

b) The coefficients are

$$a = 3, \quad b = 1+i, \quad c = 2.$$

Their magnitudes are

$$|a|^2 = 9, \quad |b|^2 = (1+i)(1-i) = 2, \quad |c|^2 = 4.$$

The total weight is

$$|a|^2 + |b|^2 + |c|^2 = 9 + 2 + 4 = 15.$$

Since $a$ corresponds to momentum $+\hbar k_1$, the probability is

$$P(p = +\hbar k_1) = \frac{|a|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{9}{15} = 0.6.$$

Note: The $A$ coefficient does not matter as it will cancel out since each magnitude will have an $A^2$.

14. A particle's wave function is given by

$$\Psi(x) = A\Big(e^{-i\theta}\, e^{-ik_1 x} + (2e^{i\phi})\, e^{ik_2 x} + (8\cos\alpha + 8i\sin\alpha)\, e^{-ik_2 x}\Big)$$

a) List the possible momentum eigenvalues of the particle.

b) Find the corresponding possible energy eigenvalues.

c) What is the probability of measuring the energy

$$E = \frac{(\hbar k_2)^2}{2m}\,?$$

a) The momentum eigenvalues correspond to the exponential terms:

$$p = -\hbar k_1, \quad p = +\hbar k_2, \quad p = -\hbar k_2.$$

b) The corresponding energy eigenvalues are

$$E_{-k_1} = \frac{(\hbar k_1)^2}{2m}, \quad E_{+k_2} = \frac{(\hbar k_2)^2}{2m}, \quad E_{-k_2} = \frac{(\hbar k_2)^2}{2m}.$$

c) The coefficients are

$$a = e^{-i\theta}, \quad b = 2e^{i\phi}, \quad c = 8\cos\alpha + 8i\sin\alpha.$$

Their magnitudes are

$$|a|^2 = |e^{-i\theta}|^2 = 1, \quad |b|^2 = |2e^{i\phi}|^2 = 4, \quad |c|^2 = |8\cos\alpha + 8i\sin\alpha|^2 = |8e^{i\alpha}|^2 = 64.$$

The total weight is

$$|a|^2 + |b|^2 + |c|^2 = 1 + 4 + 64 = 69.$$

Since both $b$ and $c$ correspond to the same energy $E = \frac{(\hbar k_2)^2}{2m}$, the probability is

$$P(E = \tfrac{(\hbar k_2)^2}{2m}) = \frac{|b|^2 + |c|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{4 + 64}{69} = \frac{68}{69} \approx 0.99.$$