Q3: A body of mass M has an external gravitational potential that can be approximated by the formula

$$\phi(x, y, z) = -\frac{GM}{R} - GQ\frac{(2z^2 - x^2 - y^2)}{4R^5},$$

where $R = \sqrt{x^2 + y^2 + z^2}$ is the distance from the center of mass of the planet and Q is the mass quadrupole moment

$$Q \stackrel{\text{def}}{=} \int \rho(x, y, z)(2z^2 - x^2 - y^2)d[\text{Vol}].$$

Here $\rho(x, y, z)$ is the mass density

Consider, for example, a spherical planet such as Neptune or Uranus that has mass M_{planet} and is encircled by a thin ring of radius a and density

$$\rho_{\text{ring}}(r, z) = m_{\text{ring}}\delta(r - a)\delta(z)/(2\pi a),$$

where $\delta(\ldots)$ is the Dirac delta function and $r = \sqrt{x^2 + y^2}$ and z are cylindrical coordinates.

a) Evaluate the contribution of both the planet and the ring to Q and M, and hence find the gravitational potential of the combined system of planet and ring.

b) Obtain the equation of motion in cylindrical polar coordinates (r, θ, z) for a small particle of mass μ that is moving in this combined potential. If the particle is initially moving in the $z = 0$ plane, will it remain in this plane?

c) Find an expression for the z-component L_z/μ of the angular momentum per unit mass of the particle and show that it is conserved.

d) Given that the particle has angular momentum per unit mass $L = L_z/\mu$, find a condition involving M, m_{ring}, a, L, and G, that must be satisfied if a circular orbit in the plane of the ring is to be possible.

e) Show that if the condition *is* satisfied, then there are *two* possible orbital radii, only one of which is stable. (**Hint**: Perhaps draw a graph of a suitable effective potential.)