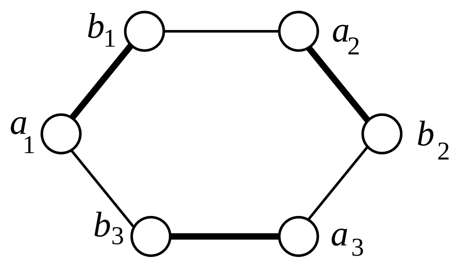
Q3:



Consider a tight-binding model of a single electron hopping on a ring of six atoms a_i, b_i, $i = 1, 2, 3$, each with a single state denoted by $|a_i\rangle$, or $|b_i\rangle$, $i = 1, 2, 3$. The ring is *dimerized* in that the bonds joining the atoms are alternately weak and strong, so that the hopping matrix elements also alternate in magnitude:

$$\begin{aligned} \langle a_1|H|b_1\rangle = \quad \langle a_2|H|b_2\rangle &= \langle a_3|H|b_3\rangle = s, \\ \langle b_1|H|a_2\rangle = \quad \langle b_2|H|a_3\rangle &= \langle b_3|H|a_1\rangle = n, \end{aligned}$$

where $s > n$ are real numbers. All other matrix elements are zero.

Define a translation operator T that acts as

$$T|a_1\rangle = |a_2\rangle, \quad T|a_2\rangle = |a_3\rangle, \quad T|a_3\rangle = |a_1\rangle,$$

and similarly for the $|b_i\rangle$'s.

a) Show that T and H commute. You can do this without working directly with the matrix representations of T and H.

b) Observe that some power of T is the identity. Use this fact to find the eigenvalues and corresponding eigenvectors $|t_i\rangle$ of T — each eigenvector as a linear combination of only the $|a_i\rangle$ or only the $|b_i\rangle$.

c) Write the non-zero Hamiltonian matrix elements in the basis of the eigenvectors of the T operator as $\langle t_i|H|t_j\rangle = \ldots.$

d) Find the eigenvalues of H. You will find this easiest by working in the eigenbasis of T. This should require diagonalizing matrices no larger than 2-by-2.