Center for Academic Resources in Engineering (CARE)
Peer Exam Review Session

Phys 213 — University Physics: Thermal Physics

Quiz 2 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Tuesday, April 22nd from 7-9pm in 4035 CIF - Aparna, Sarah, Zaahi

Can't make it to a session? Here's our schedule by course:

`https://care.grainger.illinois.edu/tutoring/schedule-by-subject`

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/844
2. Click "New Question"
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Good luck with your exam!

1. Consider a sealed container with a volume of 1 m^3 filled with 10^{23} helium atoms and 2×10^{23} molecules of nitrogen. Recall that helium is a monatomic gas and nitrogen is a diatomic gas. Initially the temperature of the gas mixture is 1000 K.

(a) Find the pressure inside the container (Answer in Pa).

A) 8980
B) 8290
C) 4140
D) 2250

(b) Find the ratio of the total molecular rotational energy to the total translational energy (molecular and atomic) inside the container. Assume that equipartition applies.

A) 0.333
B) 0.444
C) 0.666
D) 0.777

(a) The answer is **(C)**. This container holds a mixture of two gases, so the ideal gas law has to be used on each separately, and then sum of the pressures will be the total pressure from the gas mixture.

Helium	Nitrogen
$p_{He} = \frac{N_{He} k_b T}{V}$	$p_N = \frac{N_N k_b T}{V}$

Given these relations, the total pressure is the sum of each partial pressure. Note that the volume and temperature is the same for these two gases.

$$p = p_{He} + p_N = \frac{(N_{He} + N_N) k_b T}{V}$$

(b) The answer is **(B)**. First recall that the formula for the internal energy of a gas as a result of N_{DOF} degress of a freedom is $U = \frac{N_{\text{DOF}}}{2} N k_b T$. The total translational energy is the sum of the energy from the He atoms and the N atoms, both of which have 3 translational degrees of freedom per particle. Let T be the total translational kinetic energy, then

$$T = T_{He} + T_N = \frac{3}{2} N_{He} k_b T + \frac{3}{2} N_N k_b T$$

The total rotational energy, unlike the translational energy, comes entirely from the N atoms since they're diatomic. There are only 2 degrees of freedom here. Let R be the rotational kinetic energy, then

$$r = N_N k_b T$$

Taking the ratio of r to T, $\frac{r}{T}$, gives

$$\frac{r}{T} = \frac{N_N k_b T}{\frac{3}{2} N_{He} k_b T + \frac{3}{2} N_N k_b T} = \frac{2}{3} \frac{N_N}{N_{He} + N_N}$$

Note that this took into account that Helium is monatomic ($N_{\text{DOF}} = 3$), and Nitrogen is diatomic ($N_{\text{DOF}} = 5$, split among translation and rotation).

2. Which conditions are held constant in the following processes? (Refer to the formula sheet if you are not sure!)

a) Isothermal b) Isobaric c) Isochoric d) Adiabatic

a) constant temperature
b) constant pressure
c) constant volume
d) no heat added, no change in entropy

3. What is the relationship between volume and pressure during isothermal and adiabatic processes for an ideal gas, respectively?
In an isothermal process, the temperature is constant and we can apply the ideal gas law: $pV = Nk_bT =$ constant (the ideal gas law is also used for isobaric and isochoric processes).

In an adiabatic process, $pV^\gamma =$ constant where $\gamma = \frac{C_p}{C_v} = (D+2)/D$. Where C_v is the constant volume heat capacity, C_p is the constant pressure heat capacity, and D is the number of degrees of freedom. Since no heat is added (adiabatic) but work can still be done, $\Delta S = 0$. As the gas compresses, the temperature increases, and as the gas expands, the temperature decreases.

4. The following two questions refer to the setup described below.

A piston of volume 0.05 m^3 contains 5 moles of a monatomic ideal gas at 300 K. If it undergoes an isothermal process and expands until the internal pressure matches the external pressure, $P_E = 1$ atm.

(i) How much work is done by the gas on the environment?
   a) 7.42×10^3
   b) 1.12×10^4
   c) -1.12×10^4
   d) 1.83×10^4
   e) -1.83×10^4

(ii) Suppose that the piston undergoes an adiabatic expansion instead, what is the final volume of the piston, V_f? (Values have units of cubic meters)
   a) 0.086
   b) 0.095
   c) 0.123

(i) The answer is **(b)**. The work done ON the gas is $\mathrm{d}W_{on} = -p\mathrm{d}V$ while work done BY the gas is $\mathrm{d}W_{by} = p\mathrm{d}V$. They are different by a minus sign, so make sure to check your signs. To find the total work, we integrate both sides of the equation, writing p in terms of V using the ideal gas law

$$W_{by} = \int_{V_i}^{V_f} p\, dV = \int_{V_i}^{V_f} \frac{nRT}{V}\, dV = nRT \ln\left(\frac{V_f}{V_i}\right)$$

The final volume can be found using the ideal gas law (make sure pressure is in Pascals).

$$V_f = \frac{nRT}{P_E} = 0.123$$

(ii) The answer is **(a)**. The P-V relation in an adiabatic process follows $pV^\gamma = $ constant. Since the ideal gas is monatomic, $\gamma = \frac{5}{3}$. We can find the initial pressure using ideal gas law

$$P_i = \frac{nRT}{V_i} = 249420 \text{ Pa}$$

which allows us to calculate the constant $PV^\gamma = 249420 \times 0.05^{\frac{5}{3}} = 1693$

After the expansion, we can use the $p - V$ relation again and plug in $P_f = P_E$ to get

$$P_E V_f^\gamma = 1693 \rightarrow V_f = 0.086$$

5. A monatomic, ideal gas is contained at fixed volume under pressure p. Now suppose the pressure is tripled. What is the ratio of the initial v_{rms} to the final v'_{rms}? (v_{rms}/v'_{rms})
In order to calculate v_{rms}, we must equate kinetic energy with the internal energy (of a single particle of the gas).

$$\frac{1}{2}mv_{rms}^2 = \frac{3}{2}k_bT = \frac{3}{2}\frac{pV}{N}$$

Where the ideal gas law was used to make the last equation. Solving for v_{rms}, we get $\sqrt{\frac{3pV}{Nm}}$

If we then triple the pressure, we get $v'_{rms} = \sqrt{\frac{9pV}{Nm}}$

Taking the ratio, $\frac{v_{rms}}{v'_{rms}}$, we get

$$\frac{v_{rms}}{v'_{rms}} = \frac{\sqrt{\frac{3pV}{Nm}}}{\sqrt{\frac{9pV}{Nm}}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

6. Using the second law of thermodynamics, show that it is impossible for a heat engine to operate at $\epsilon = 1$.
Start with the fact that $\Delta S_{total} = \Delta S_H + \Delta S_C$ (the total change in entropy is the sum of the change in entropy of the hot and cold reservoir).

We know from the definition of temperature that $\frac{1}{T} = \frac{\mathrm{d}S}{\mathrm{d}U}$. This lets us rewrite the above as

$$\Delta S_{total} = \frac{\Delta U_H}{T_H} + \frac{\Delta U_C}{T_C} = -\frac{Q_H}{T_H} + \frac{Q_C}{T_C}$$

Where the last step is a result of $Q = \Delta U$ since the reservoirs do no work. Now using the second law, $\Delta S_{total} \geq 0$, we can see $\frac{Q_C}{Q_H} \geq \frac{T_C}{T_H}$

Combining this with the definition of engine efficiency: $\epsilon = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}$

We can see the that there is a limit on ϵ

$$\boxed{\epsilon \leq 1 - \frac{T_C}{T_H}}$$

7. Consider a balloon of volume 3 m^3 and at an initial pressure of 3 atmospheres.

(a) If the pressure is held constant and the balloon is heated up, what is the work done by the balloon on its environment given that the balloon expands to 5 m^3?

(b) If the volume is held constant, what is the new work done on the environment given that the balloon is heated up?

(c) If the pressure is not constant but now a function of volume, what is the work done on the environment given that the balloon is at 387 K and contains 283 moles of a monatomic gas, and expands to 5 m^3?

(a) If pressure is held constant, then we can pull it out of the integral for work.

$$W = \int_3^5 p dV$$

$$p = 3\text{atm} \approx 304,000\text{Pa}$$

Thus:

$$W = 304,000V|_3^5 = 608\text{kJ}$$

(b) If Volume is held constant, no work can be exerted and thus W = 0

(c) If $p = p(V)$, then we can invoke the ideal gas law to substitute out p!

$$W = \int_3^5 p dV = \int_3^5 \frac{nRT}{V} dV$$

Now we can integrate.

$$W = nRT \int_3^5 \frac{1}{V} dV = nRT \ln\left(\frac{5}{3}\right) = 465.136\text{kJ}$$

The decrease in energy in problem (c) relative to problem (a) is because the pressure can change in (c)! A loss of pressure indicates less energy in the air molecules within the balloon, which translates to less energy transferred to the environment as the balloon inflates.

8. A Carnot heat engine (one operating at maximum efficiency) is operating between two reservoirs at T_H and T_C. Suppose that 1 kJ of heat must be added to the engine in order to produce 500 J of work and you measured the temperature of the cold reservoir to be 300 K, what is the temperature of the hot reservoir?

   a) 150 K
   b) 200 k
   c) 300 K
   d) 450 K
   e) 600 K

   The answer is **(e)**. The efficiency of this particular engine is $\epsilon = \frac{W}{Q_H} = \frac{500\text{ J}}{1\text{ kJ}} = \frac{1}{2}$. However, we are told that this is a Carnot engine, meaning it is operating at its maximum possible efficiency, which is given by

$$\epsilon_{max} = 1 - \frac{T_C}{T_H}$$

   Knowing that the observed efficiency is the maximum efficiency (by definition of a Carnot engine), we can equate

$$\epsilon = \frac{1}{2} = 1 - \frac{T_C}{T_H}$$

   Solving for the unknown value of T_H, we find 600K.

9. A heat pump uses 200 J of work to remove 300 J of heat from a cold reservoir. How much heat would be delivered to the hot reservoir?
   The relationship between the work done, the energy extracted form the cold reservoir, and the energy added to the hot reservoir is

$$Q_H = W + Q_C$$

   This is also the equation used for a heat engine when all the energy originates from the hot reservoir. You can see this is true from conservation of energy.

   Using this fact, and remembering that all these quantities are defined to be positive, we get

$$\boxed{Q_H = 500\text{ J}}$$

10. The second law of thermodynamics states that the change in entropy is always greater than or equal to zero, yet sometimes in our calculations we can get a negative change in entropy for a piece of our system. What does a negative change in entropy represent? And how do negative changes not violate the second law?

A negative change in entropy means that the object is transitioning to a state with fewer microstates (which can be thought of as a less "diverse" state).

The second law states that the change in entropy *for the entire system considered* must be greater than or equal to zero, thus we can have individual components of a larger system that decrease in entropy, but that would require another part of the system gaining at least that much entropy.

11. Suppose we have some ocean water (which we will treat as pure water for our purposes) connected to an atmosphere. The atmosphere above contains an abundance of CO2 gas. CO2 can dissolve into water just like a solid. What is true about the total Gibbs Free Energy and Internal Energy of the ocean water system after enough CO2 has dissolved to come to equilibrium, acidifying the ocean?

    Don't let the scenario distract you from what's really going on! At the end of the day, the Gibbs Free Energy will be minimized when the system comes into equilibrium to maximize entropy. The Internal Energy will increase because accepting a molecule of CO2 requires energy (specifically, to break the hydrogen bonds between adjacent water molecules).

12. How do you calculate the work from an isothermal process?
    Work from any thermodynamic process is defined as:

$$W = \int_{V_i}^{V_f} p dV$$

But wait! p changes as V changes, so we can't take the integral naively. If we assume an ideal gas, i.e.:

$$pV = Nk_bT \implies p = \frac{Nk_bT}{V}$$

Now we have parameterized p as a function of V! Moreover, Nk_bT is a constant under an isothermal process, so we can pull it out of the integral. We are left with:

$$W = \int_{V_i}^{V_f} \frac{Nk_bT}{V} dV \implies W = Nk_bT \int_{V_i}^{V_f} \frac{1}{V} dV$$

Evaluating the integral, we get:

$$W = Nk_bT(\ln V_f - \ln V_i) = Nk_bT \ln \frac{V_f}{V_i}.$$

13. How does a heat engine's work output and efficiency scale with the temperatures of its reservoirs? If you were to design a perfect heat engine, what would be the lowest temperature you should set your cold reservoir to?
    The efficiency and work output increase linearly with the difference in temperatures between the two reservoirs.

$$\epsilon = \frac{W}{Q_H} = \frac{Q_H - Q_C}{Q_H}$$

Since the difference $Q_H - Q_C$ increases as the temperature between the reservoirs increases (heat scales proportionally with temperature) we know that the work, and thus the efficiency of an engine, increases as well from the math.
Ideally, one would set the cold reservoir's temperature to 0 K (absolute zero) to get the maximum temperature difference, and hence the maximum amount amount of work out (and an efficiency of 100%). However, since true absolute zero is not attainable, we must restrict ourselves to the Carnot efficiency.

$$\epsilon \leq \frac{T_H - T_C}{T_H}$$

The problem is the lower you set T_C to be, the more energy you need to put into a **heat pump** to lower your reservoir to that desired temperature. At some point, the costs of the energy expenditures outweigh the benefit of using the heat pump, and this is why our engines are restricted to always be far below the Carnot efficiency.

14. True or False: For an ideal gas, the molar heat capacity at constant pressure will always be greater than the molar heat capacity at constant volume.
From the definition of molar heat capacity at constant pressure

$$c_{p,mol} = \frac{1}{n}\frac{\mathrm{d}U}{\mathrm{d}T} + \frac{p}{n}\frac{\mathrm{d}V}{\mathrm{d}T}$$

and the definition of molar heat capacity at constant volume

$$c_{v,mol} = \frac{1}{n}\frac{\mathrm{d}U}{\mathrm{d}T}$$

we can relate the two by substitution, ending up with

$$c_{p,mol} = c_{v,mol} + \frac{p}{n}\frac{\mathrm{d}V}{\mathrm{d}T}$$

So we already see that $c_{v,mol}$ is less than $c_{p,mol}$ as long as $\frac{p}{n}\frac{\mathrm{d}V}{\mathrm{d}T} > 0$. Now let's confirm that, for an ideal gas, $\frac{p}{n}\frac{\mathrm{d}V}{\mathrm{d}T}$ is greater than zero.
Starting from the ideal gas law $pV = Nk_bT$, we can differentiate V with respect to T to get $p\frac{\mathrm{d}V}{\mathrm{d}T} = Nk_b$. Substituting this into the equation relating the two molar heat capacities and simplifying, we get

$$\begin{aligned} c_{p,mol} &= c_{v,mol} + \frac{p}{n}\frac{\mathrm{d}V}{\mathrm{d}T} \\ c_{p,mol} &= c_{v,mol} + \frac{N}{n}k_b \\ c_{p,mol} &= c_{v,mol} + N_A k_b \\ c_{p,mol} &= c_{v,mol} + R \end{aligned}$$

Note that $\frac{N}{n}$ is Avogadro's Number, N_A, and R is the universal gas constant. Since $R > 0$, we have confirmed that $c_{p,mol} > c_{v,mol}$ for an ideal gas.

$$\boxed{\text{True}}$$

15. For this problem, we'll be playing around with differentials.

(a) Recall that the expression for Gibbs Free Energy is

$$G = U - TS + pV$$

Without making any substitutions, write out the full differential form of G, $\mathrm{d}G$.

(b) Determine the derivative of G with respect to volume V at fixed pressure, temperature, and number of particles. That is, find

$$\left(\frac{\partial G}{\partial V}\right)_{T,\ p,\ N}$$

and simplify as much as you can.

(c) Should G be minimized or maximized at equilibrium? What does this mean $\partial G/\partial V$ should be equal to at equilibrium?

(d) Determine a formula for the equilibrium volume V_{eq} given that this gas is not ideal, i.e.

$$S = Nk\ln(V - bN)$$

(e) Determine the equilibrium volume if $T = 340$ K, $p = 50$ kPa, $n = 4$ mol, and $b = 8.1 \times 10^{-27}$ m^3.

(a) To find the differential, we must apply chain rule to the other variables:

$$\begin{aligned} G &= U - TS + pV \\ \mathrm{d}G &= \mathrm{d}U - S\mathrm{d}T - T\mathrm{d}S + V\mathrm{d}p + p\mathrm{d}V \end{aligned}$$

(b) First, we divide the expression we found in (a) by the differential $\mathrm{d}V$:

$$\begin{aligned} \frac{\partial G}{\partial V} &= \frac{\partial U}{\partial V} - S\frac{\partial T}{\partial V} - T\frac{\partial S}{\partial V} + V\frac{\partial p}{\partial V} + p\frac{\partial V}{\partial V} \\ &= \frac{\partial U}{\partial V} - S\frac{\partial T}{\partial V} - T\frac{\partial S}{\partial V} + V\frac{\partial p}{\partial V} + p \end{aligned}$$

Then, we fix the variables to be fixed: T, p, and N. To deal with them, we set any derivative of these variable to 0:

$$\begin{aligned} \left(\frac{\partial G}{\partial V}\right)_{T,\ p,\ N} &= \frac{\partial U}{\partial V} - S(0) - T\frac{\partial S}{\partial V} + V(0) + p \\ &= \frac{\partial U}{\partial V} - T\frac{\partial S}{\partial V} + p \end{aligned}$$

Finally, recall that $U = \frac{N_{\text{DOF}}}{2}NkT$. In other words, U depends on T and N, but since these variables are fixed, U is also fixed, so derivatives of U are equal to 0.

$$\left(\frac{\partial G}{\partial V}\right)_{T,\ p,\ N} = -T\frac{\partial S}{\partial V} + p$$

(c) G should be *minimized* at equilibrium. Thus, its derivative with respect to volume should be 0 at equilibrium.

(d) We can plug this S equation into the expression we found in (b):

$$\begin{aligned}\left(\frac{\partial G}{\partial V}\right)_{T,\ p,\ N} &= -T\frac{\partial}{\partial V}\left(Nk\ln(V-bN)\right)+p\\ &= -T\left(\frac{Nk}{V-bN}\right)+p\end{aligned}$$

G should be minimized, so we set $\partial G/\partial V$ equal to 0:

$$\begin{aligned}0 &= -T\left(\frac{Nk}{V-bN}\right)+p\\ p &= \frac{NkT}{V-bN}\\ pV-pbN &= NkT\\ V &= \frac{NkT+pbN}{p}\end{aligned}$$

We can also maximize Entropy (S) instead of minimizing Gibbs Free Energy (G), because they are the same thing:

$$S = Nk\ln(V-bN)$$

$$\frac{\partial S}{\partial V} = \frac{p}{T} = \frac{Nk}{V-bN}$$

$$V = \frac{NkT+pbN}{p}$$

We get the same equation for V.

Thus, maximizing Entropy is the same as minimizing Gibbs Free Energy.

(e) We simply plug the given quantities into the equation we found in (d). Note, however, that we're given the number of *moles* which must be converted into the number of molecules ($N = nN_A$). We get $V = 0.246\ \text{m}^3$.

16. Maximizing the Entropy of the system and environment is equivalent to:

(a) Minimizing the Gibbs Free Energy of the system
(b) Minimizing the Gibbs Free Energy of the system and environment
(c) Minimizing Temperature
(d) Maximizing Work

Answer: a
Consider the Second Law of Thermodynamics:

$$dS_{tot} = dS_{sys} + dS_{res} > 0$$

The thermodynamic identity says

$$dS_{res} = \frac{dU_{res}}{T} + \frac{p}{T}dV_{res} \implies dS_{res} = -\frac{dU_{sys}}{T} - \frac{p}{T}dV_{sys}$$

since a change in the energy/volume of the reservoir is the same as a negative change in the energy/volume of the system. Plugging this into dS_{tot}, we find:

$$dS_{tot} = dS_{sys} - \frac{dU_{sys}}{T} - \frac{p}{T}dV_{sys} > 0$$

$$dS_{tot} = \frac{-1}{T}(dU_{sys} + pdV_{sys} - TdS_{sys}) > 0 \implies dS_{tot} = -\frac{dG_{sys}}{T} > 0$$

Which shows that maximizing the total entropy of the system equates to minimizing the system's Gibbs Free Energy.