The Grainger College of Engineering
Center for Academic Resources in Engineering

MATH 241

Midterm 3 Review

Keep in mind that this presentation was created by CARE tutors, and while it is thorough, it is not comprehensive.

QR Code to the Queue

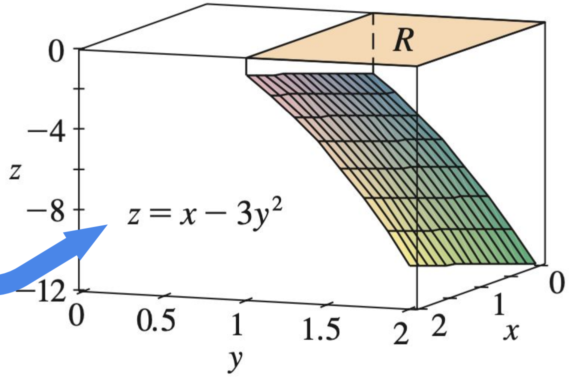
The queue contains the worksheet and the solution to this review session

Fubini's Theorem

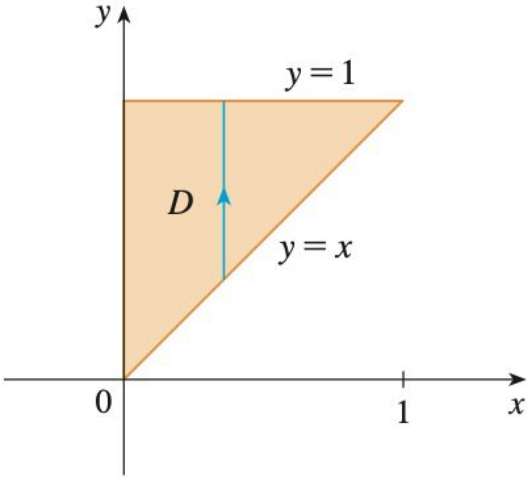
- If f(x,y) is continuous on the rectangle

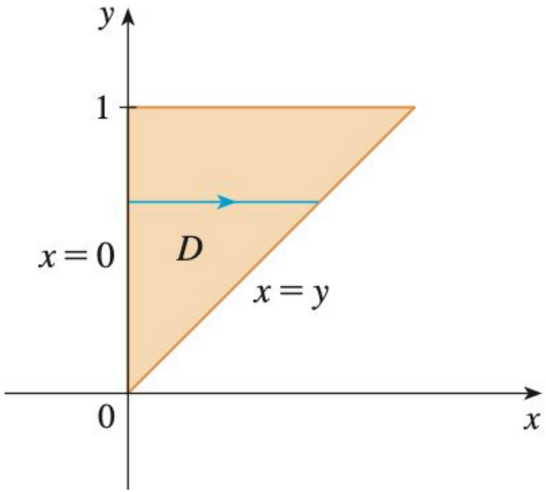
$$R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$$

$$\iint f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$



Double Integral Over a General Region





- Integrate dy from y=x to y=1
- Then integrate dx

- Integrate dx from x=0 to x=y
- Then integrate dy

Center of Mass

- The x, y coordinates of the center of mass for an object that has a density function ρ(x,y)

$$\bar{\mathrm{x}} = \frac{1}{\mathrm{m}}\iint x \cdot \rho(x, y)dA \qquad \bar{\mathrm{y}} = \frac{1}{\mathrm{m}}\iint y \cdot \rho(x, y)dA$$

, where mass is calculated as $\mathrm{m} = \iint \rho(x, y)dA$

Triple Integral

- **Let *E* be the solid contained under the plane 2x + 3y + z = 6 in the first octant. Compute the following:**

$$\iiint_E 2x \, dV$$

Triple Integral-Cont'd

- **Let *E* be the solid contained under the plane 2x + 3y + z = 6 in the first octant. Compute the following:**

$$\iiint_E 2x\,dV = \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3y} 2x\,dzdydx = \int_0^3 \int_0^{2-2x/3} 2x(6\text{-}2x\text{-}3y)dydx$$

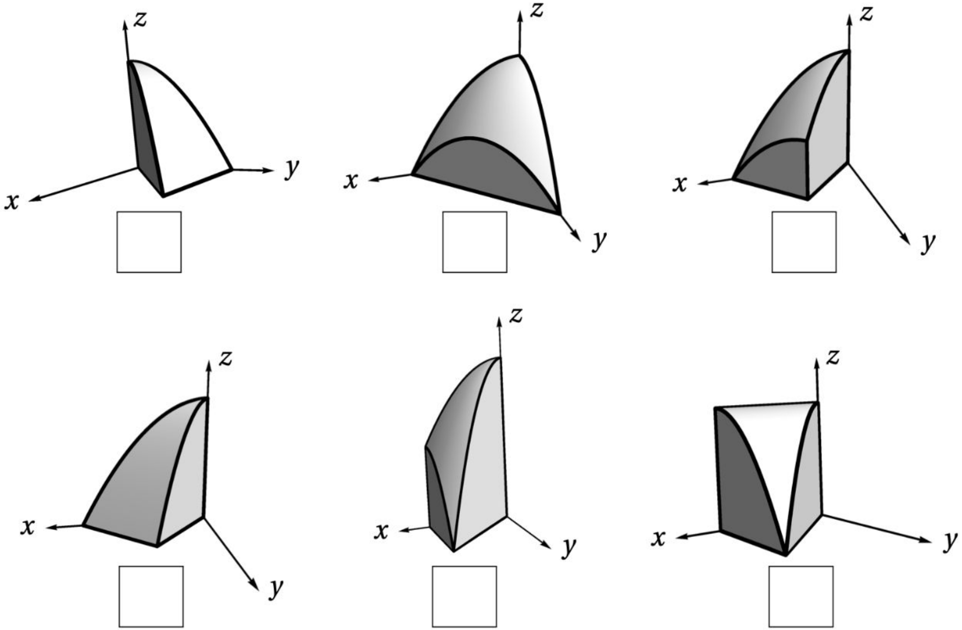
$$= \int_0^3 12x\left(2-\frac{2x}{3}\right) - 4x^2\left(2-\frac{2x}{3}\right) - 3x\left(2-\frac{2x}{3}\right)^2 dx = 9$$

Example Question #1

- **Match the integrals to their corresponding solid regions:**

(A) $\displaystyle\int_0^1 \int_y^1 \int_0^{2-x^2-y^2} f(x,y,z)\, dz\, dx\, dy$

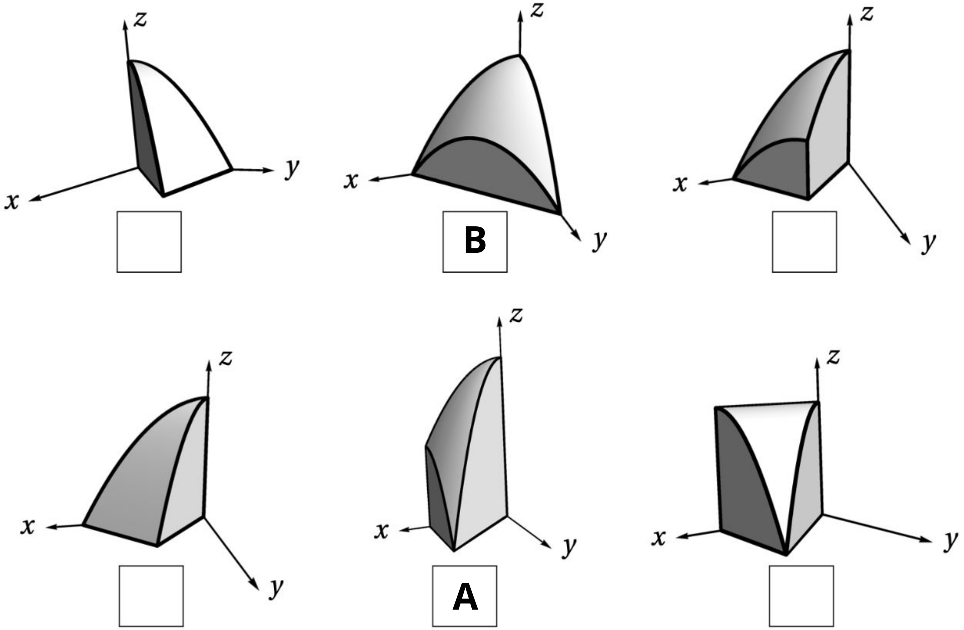
(B) $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} g(x,y,z)\, dz\, dy\, dx$



Example Solution #1

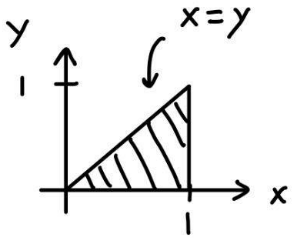
$$\text{(A)} \quad \int_0^1 \int_y^1 \int_0^{2-x^2-y^2} f(x,y,z)\, dz\, dx\, dy$$

$$\text{(B)} \quad \int_0^1 \int_0^{1-x} \int_0^{1-x^2-y^2} g(x,y,z)\, dz\, dy\, dx$$

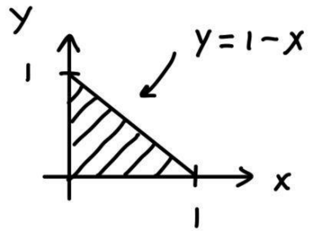


base

A



B



Polar Coordinates

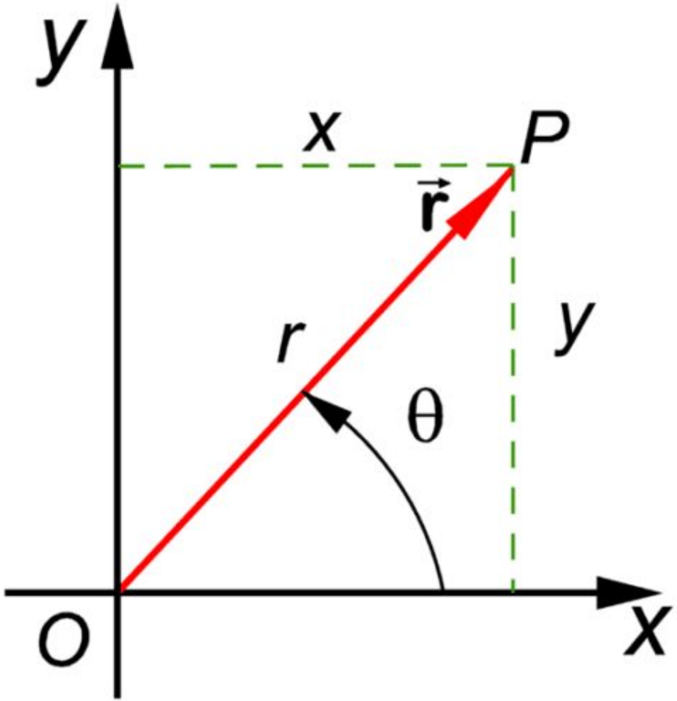
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\theta = arctan\left(\frac{y}{x}\right)$$

$$dA = rdrd\theta$$



https://magoosh.com/hs/ap-calculus/2017/ap-calculus-bc-review-polar-functions/

Cylindrical Coordinates

- **Cylindrical coordinate is just an extension of polar coordinate to three dimension**

$$x = r\cos\theta$$

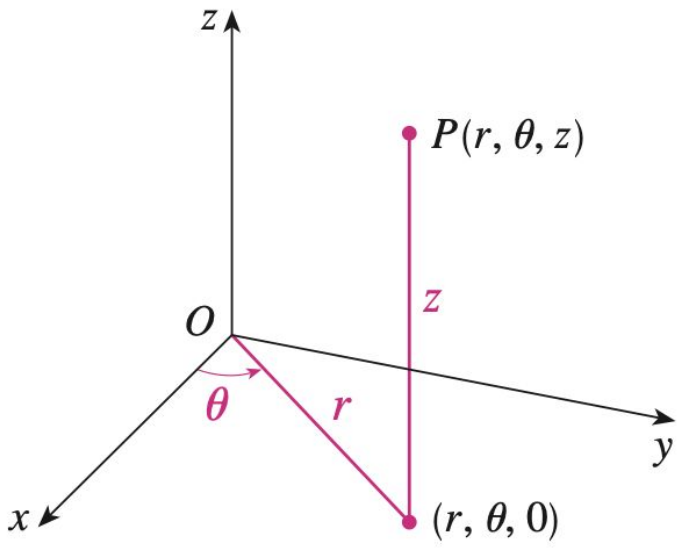
$$y = r\sin\theta$$

$$z = z$$

$$r^2 = x^2 + y^2$$

$$\theta = arctan\left(\frac{y}{x}\right)$$

$$dV = r\,dz\,dr\,d\theta$$



Sketch of a point in R^3

Spherical Coordinates

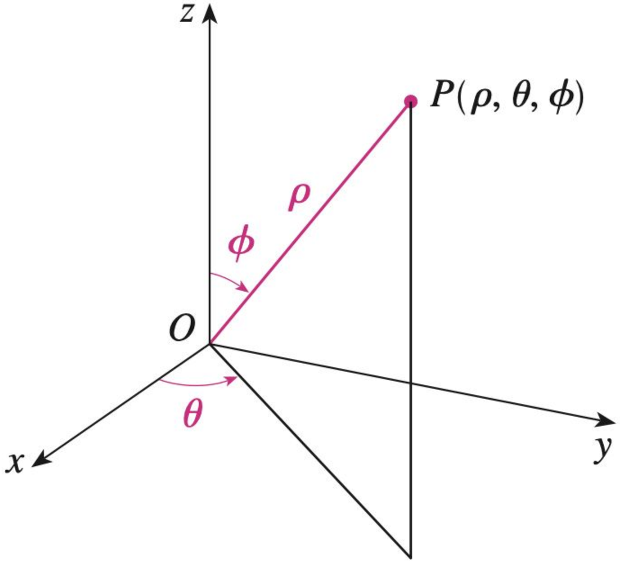
$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$$



Sketch of a point in R^3

Surface Area

- **The area of the surface A(S) with equation z=f(x,y) can be calculated as:**

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Change of Variables Using Jacobian Matrix

- **If there is a transformation such that x=g(u,v) and y=h(u,v), then:**

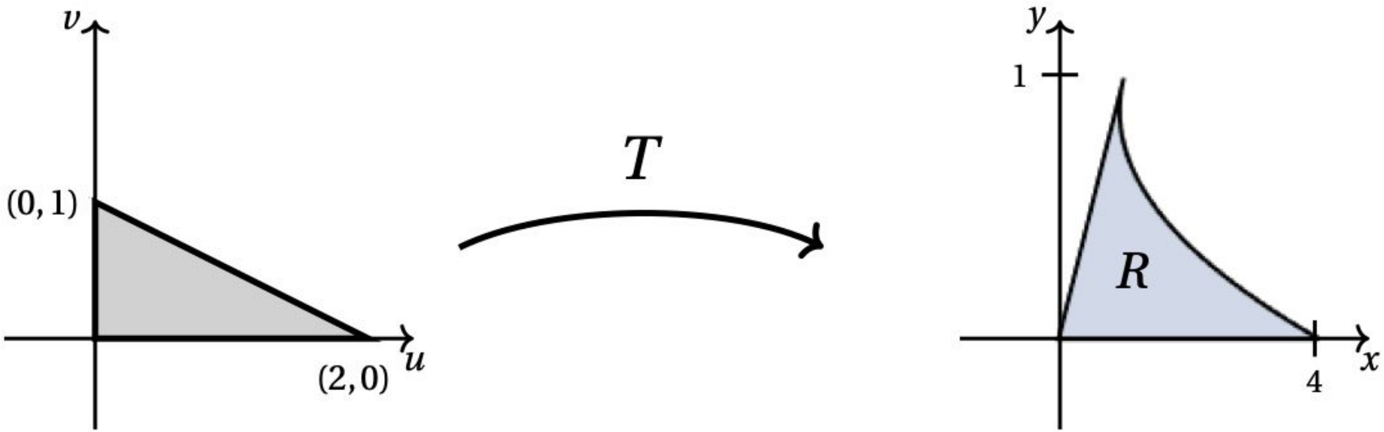
$$\iint_R f(x,y)\,dA = \iint_S f[g(u,v),h(u,v)] \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| d\bar{A}$$

, where the Jacobian Matrix is calculated as

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

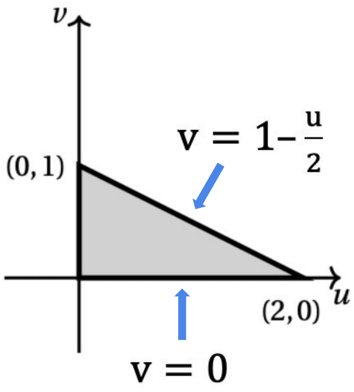
Example Question #2

- **Set up the integral to calculate the area of R with the transformation $T(u,v) = (u^2+v, v)$.**



Example Solution #2

- **Set up the integral to calculate the area of R with the transformation T(u,v) = (u^2+v, v).**



$$0 \le v \le 1 - \frac{u}{2} \qquad 0 \le u \le 2$$

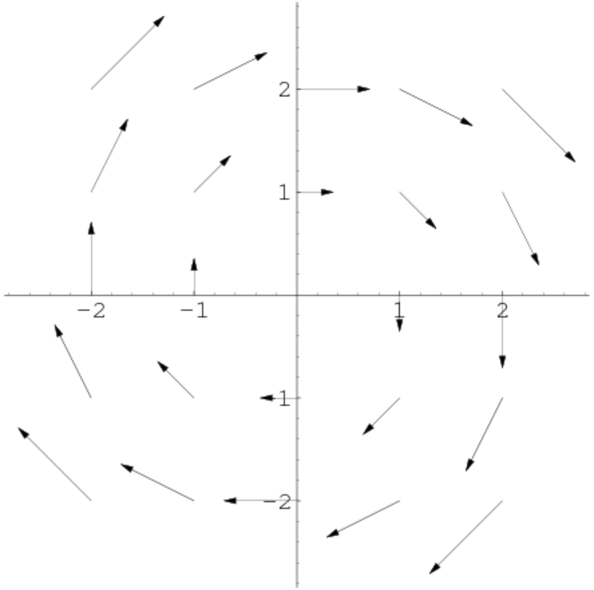
Jacobian: $\det \begin{bmatrix} 2u & 1 \\ 0 & 1 \end{bmatrix} = 2u$

Integral: $\int_0^2 \int_0^{1-u/2} 2u \, dvdu$

Vector Field

- A function that assigns a vector $\mathbf{F}$ to each point in 2D or 3D space.
- Takes in a point and “spits out” a vector

$$\vec{\mathbf{F}}(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}.$$



Conservative Vector Field

- Line integrals of a conservative vector field are independent of path

$\int_C F \cdot dr$ is independent of path D if and only if

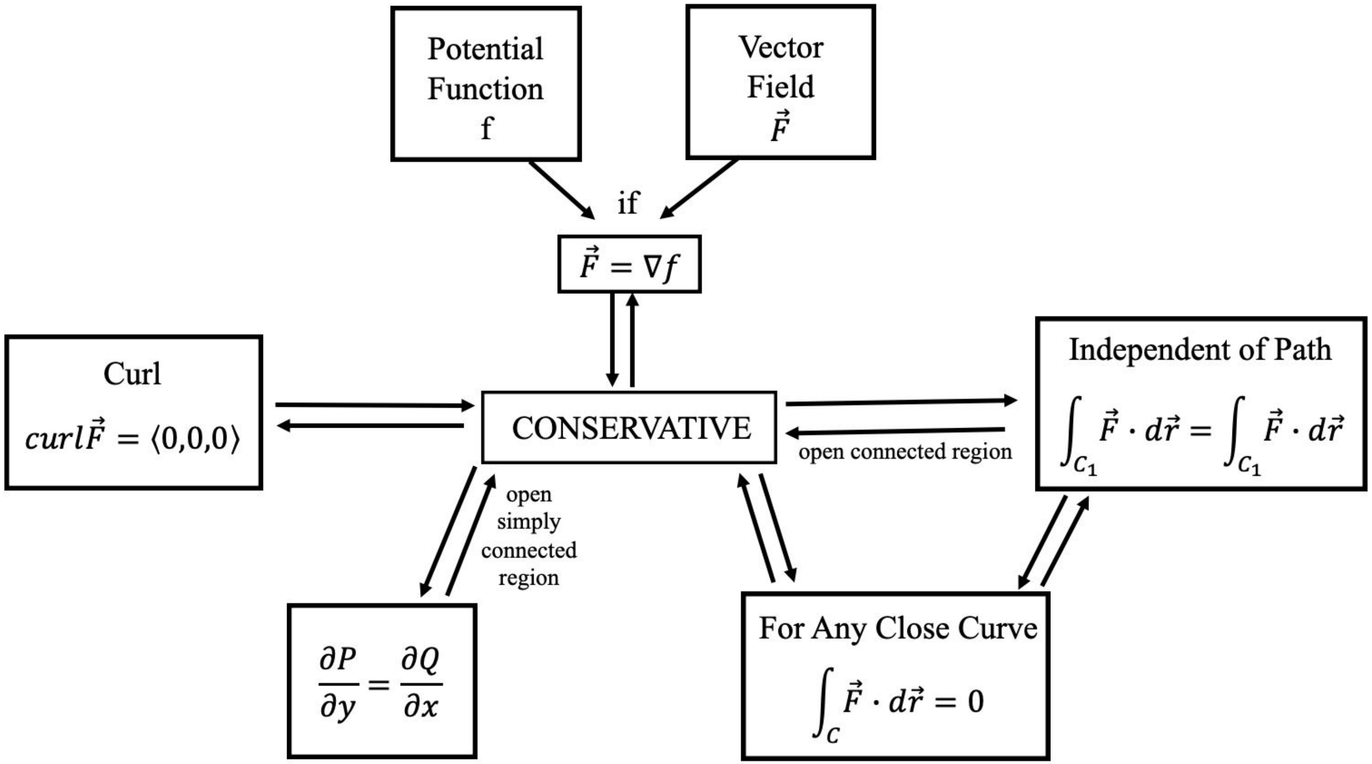
$\int_C F \cdot dr = 0$ for every closed path C in D

- Let F = P**i** + Q**j** be a vector field on an open simply-connected region D. Suppose that P and Q have continuous partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

, then F is conservative.

Conservative Vector Field



Green's Theorem

- Let C be a **counterclockwise**, **simple closed curve** in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

- Green's theorem to calculate the area of a region D bounded by C

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

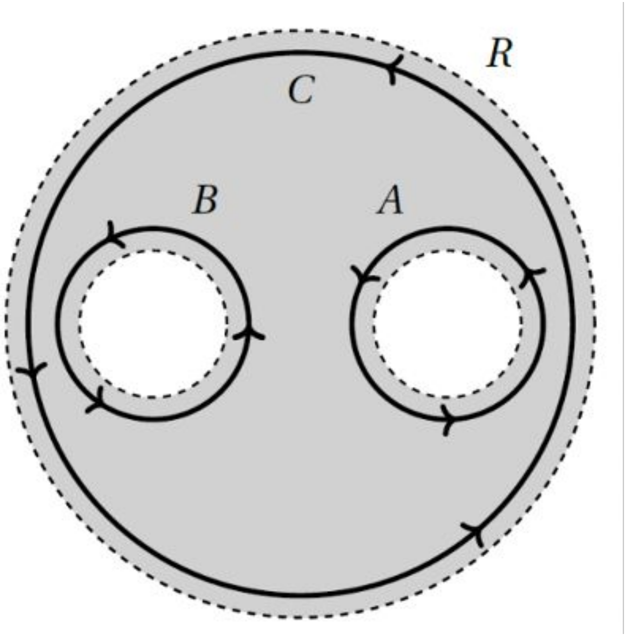
Example Question #3

- **Consider the region R shown at the right which contains simple closed curves A, B, and C. Suppose F = <P, Q> is a vector field with continuous partial derivatives on R with the following characteristics:**

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \qquad \int_A F \cdot dr = 2 \qquad \int_B F \cdot dr = -1$$

(a) Find $\int_C F \cdot dr$

(b) Is this vector field conservative?

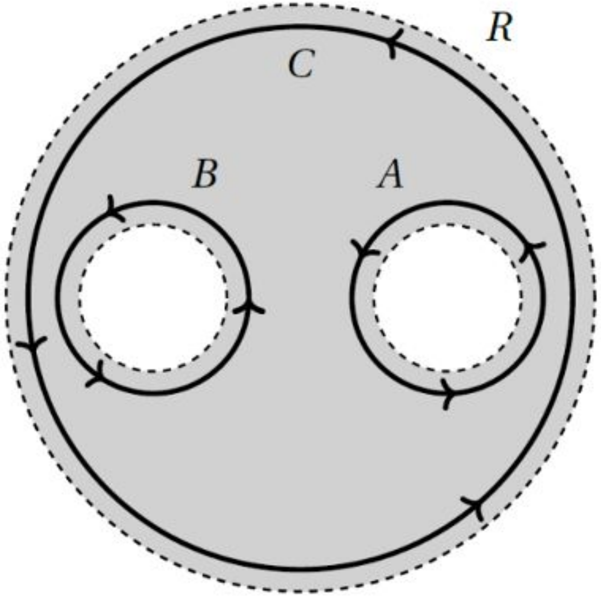


Example Solution #3

(a) Let D be the region enclosed by C. Using Green's theorem:

$$\int_C F \cdot dr - \int_A F \cdot dr - \int_B F \cdot dr = 0$$

$$\int_C F \cdot dr - 2 - (-1) = 0 \qquad \boxed{\int_C F \cdot dr = 1}$$



(b) This vector field is not conservative because it is not a simply-connected region, and the line integral for the closed curve C is not 0.

Curl

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

- Cross product → Curl is a **vector field**
- Describes how vectors **rotate** around a certain point
- Use **right-hand rule** to determine the sign of curl
- Curl of a gradient field = 0
- If F is conservative, curl = 0
- Green's theorem in vector form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA$$

Curl Test for Conservative Vector Field

- If F is a vector field defined on all of R^3 whose component functions have **continuous partial derivatives** and **curl F = 0**, then F is a conservative vector field

Divergence

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}$$

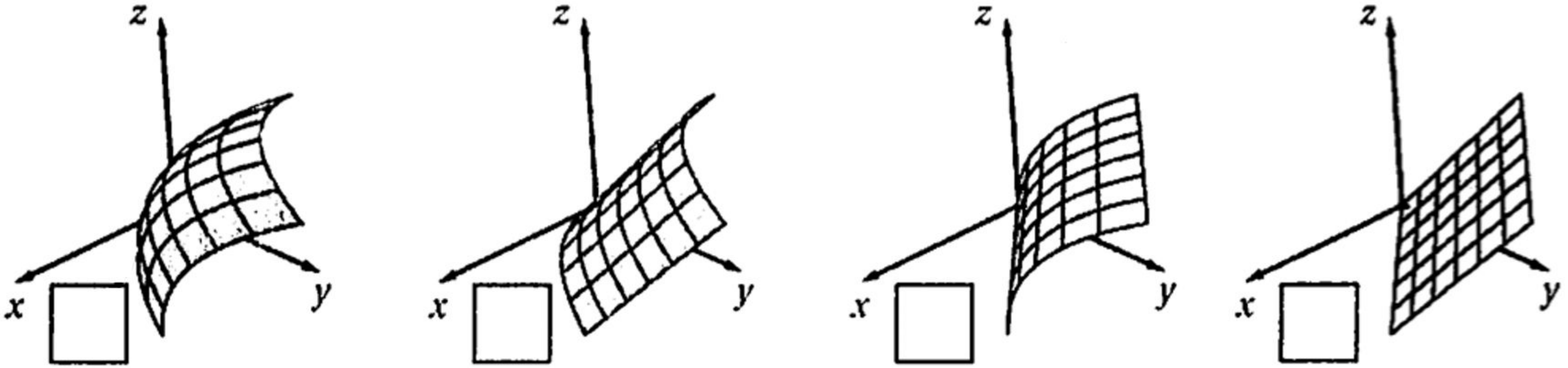
- Dot product → Divergence is a **scalar** field
- Describes how vectors diverge from a single point (or converge to a point)
- **Diverging vectors: positive, Converging vectors: negative**
- Green’s theorem in vector form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

Example Problem #4

- **Match the surfaces below with the following parametrization:**

$$r(u,v) = \langle u, u^2 + v^2, v \rangle \text{ defined on } D = \{(u,v) | 0 \le u \le 1, 0 \le v \le 1\}$$



Example Solution #4

$$r(u,v) = < u, u^2 + v^2, v > \text{ defined on } D = \{(u,v) | 0 \le u \le 1, 0 \le v \le 1\}$$

When x is constant → curve on the yz-plane should be a parabola

When y is constant → curve on the xz-plane should be a circle

When z is constant → curve on the xy-plant should be a parabola