Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

MATH 257 – Linear Algebra with Computational Applications

Midterm 3 Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: Apr. 15, 6:00 - 7:30 PM Alice, Danielle, Jash, Nehan

Session 2: Apr. 16, 6:15 - 8:15 PM Rohan, Ryan, Maanvi, Maheen

Can't make it to a session? Here's our schedule by course:

https://care.grainger.illinois.edu/tutoring/schedule-by-subject

Solutions will be available on our website after the last review session that we host.

Step-by-step login for exam review session:

1. Log into Queue @ Illinois: https://queue.illinois.edu/q/queue/955
2. Click "New Question"
3. Add your NetID and Name
4. Press "Add to Queue"

Please be sure to follow the above steps to add yourself to the Queue.

Good luck with your exam!

1. Compute the least squares solution of

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a)
$$\begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{1}{2} \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c)
$$\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$$

d)
$$\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

e) None of these

We need to solve

$$\boldsymbol{A}^T \boldsymbol{A} \hat{x} = \boldsymbol{A}^T b$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Multiplying out we get

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving, we get

$$\hat{x} = \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}, \text{ or } \boxed{c}$$

2. Suppose $\boldsymbol{B}$ is a 3×3 matrix and $\det(\boldsymbol{B}) = 5$. Determine the following quantities:

a) $\det(2\boldsymbol{B})$

Properties of determinants: multiplying the matrix by a scalar is the same as multiplying the determinant by that scalar to the nth power: c^n (where n is the dimension of the matrix).
$2^3\det(\boldsymbol{B}) = \boxed{40}$.

b) $\det(\boldsymbol{B}^T\boldsymbol{B})$

Properties of determinants: the determinant of the transpose is the same as the determinant of the matrix and the determinant of a matrix product is the product of the two determinants.

$\det(\boldsymbol{B})$*$\det(\boldsymbol{B}) = \boxed{25}$.

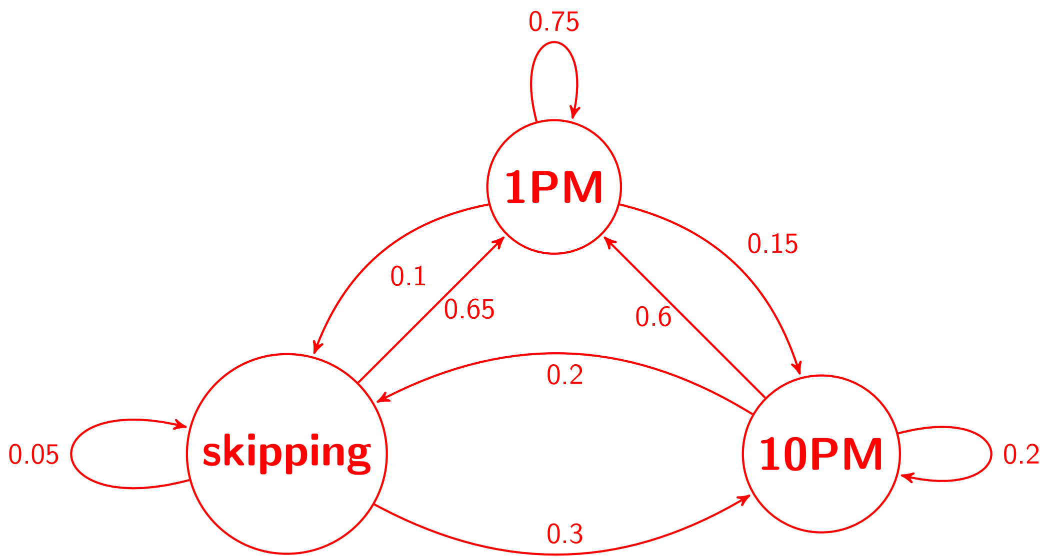
c) $\det(\boldsymbol{B}^{-1})$

Properties of determinants: the determinant of the inverse matrix is the inverse of the determinant of the original matrix.

$\det(\boldsymbol{B}^{-1}) = 1/\det(\boldsymbol{B}) = \boxed{1/5}$.

3. Every week at Monster's University the rate of students attending Scaring 101 follows this pattern: 75% of students who attended the 1PM lecture the week before come to class, 15% switch to the 10AM section, and 10% don't attend lecture at all. On the other hand, 20% of the students who went to the 10AM section the week before come back to class, 60% switch to the 1PM lecture, and 20% skip class. Finally, 65% of students who skipped class attend the 1PM lecture, 30% come to the 10AM section, and 5% skip once again.

a) Set up a Markov matrix to represent the attendance of Scaring 101 at Monster's University.
If it helps, first draw a graph based on the word problem.



Then write out the Markov matrix with column 1 as the 1PM lecture, column 2 as the 10AM lecture, and column 3 as students who skip lecture:

$$A = \begin{bmatrix} 0.75 & 0.6 & 0.65 \\ 0.15 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.05 \end{bmatrix}$$

As a sanity check, make sure the columns sum to 1!

b) If on the first week of the semester, 50% of students attend the 1PM lecture and 50% attend the 10AM lecture, how would you calculate the lecture attendance after 3 weeks?

Note that since all students attended a lecture, the percentage of students who skipped is 0%. Set up the initial probability vector $\boldsymbol{x} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$.

Then, the attendance after three weeks is $A^3\boldsymbol{x}$. This matrix is NOT nice to diagonalize, so manually calculating the product is probably better. You can also code this easily in Python (highly recommended)!

$$\text{If you want to try calculating, you should get } \begin{bmatrix} 0.7118 \\ 0.1756 \\ 0.1126 \end{bmatrix}.$$

This means that 71.18% of students attend the 1PM lecture, 17.56% attend the 10AM lecture, and 11.26% of students skip class.

c) Set up, but do not solve the matrix for finding the steady state probability vector.
The steady state vector is in the nullspace of A-I. Thus, the expression is:

$$Null(\begin{bmatrix} -0.25 & 0.6 & 0.65 \\ 0.15 & -0.8 & 0.3 \\ 0.1 & 0.2 & -0.95 \end{bmatrix}).$$

4. Given the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & -3 & 17 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the characteristic polynomial for the matrix.

$$det(A - tI_4) = \begin{bmatrix} 3-t & 0 & 0 & 0 \\ 0 & 2-t & 3 & 17 \\ 0 & 0 & 5-t & 0 \\ 0 & 0 & 0 & 1-t \end{bmatrix} = (1-t)(2-t)(3-t)(5-t)$$

a) Find the eigenvalues of A.
Solve for the roots of the characteristic polynomial: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_5 = 5$. Note that since A is a triangular matrix, the eigenvalues are just the diagonal entries.

b) Find the eigenvector, algebraic multiplicity, and geometric multiplicity for the largest eigenvalue of A.
The largest eigenvalue is $\lambda_4 = 5$. Solve for the nullspace of $A - 5I_4$:

$$Null(\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & -3 & 17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}) \xrightarrow{REF} Null(\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}) \text{ to get } span\left(\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$$

Thus, the eigenvector for $\lambda = 5$ is $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

The algebraic multiplicity is 1 (because the exponent in the characteristic polynomial is 1), and the geometric multiplicity is 1 because there is only one vector is needed to span the eigenspace of 5.

5. Given the matrix

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$$

a) Calculate the diagonalization of A, PDP^{-1}.
First find the eigenvalues of A. The characteristic polynomial is $(6-\lambda)(3-\lambda)+2 = \lambda^2 - 9\lambda + 20 = (\lambda - 5)(\lambda - 4)$. Thus, the eigenvalues are 4 and 5. Next, find spanning eigenvectors for both.

$$\lambda = 4\text{: Null(A-4I) is spanned by } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\lambda = 5\text{: Null(A-5I) is spanned by } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\text{That means PD}P^{-1} = \boxed{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}}$$

b) Calculate e^{At}.

$$e^{At} = \text{P}e^{Dt}P^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

$$\text{Use } \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ to calculate the inverse and evaluate:}$$

$$\begin{bmatrix} e^{4t} & e^{5t} \\ 2e^{4t} & e^{5t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} -e^{4t} + 2e^{-5t} & e^{4t} - e^{5t} \\ -2e^{4t} + 2e^{5t} & 2e^{4t} - e^{5t} \end{bmatrix}}$$

c) Suppose the following is a system of linear ordinary differential equations:

$$\frac{du_1}{dt} = 6u_1 - u_2$$
$$\frac{du_2}{dt} = 2u_1 + 3u_2$$

What is the general solution? (Hint: use your work from the previous part).
We already have the eigenvalues and eigenvectors, so the general solution is $c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_1 t} v_2$:

$$\mathbf{u} = \boxed{c_1 e^{4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

d) Find the particular solution for the previous system if the initial condition is $\mathbf{u}(0) = \begin{bmatrix} 13 \\ 21 \end{bmatrix}$.

Plug in t = 0 and get that $e^0 = 1$. Thus, the general solution reduces to:

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solve the problem with a matrix:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 21 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

So the particular solution is $\boxed{8e^{4t}\begin{bmatrix}1\\2\end{bmatrix} + 5e^{5t}\begin{bmatrix}1\\1\end{bmatrix}}$

6. Given an unknown 2x2 matrix A where det(A) = 5, and a 3x3 matrix

$$B = \begin{bmatrix}-2 & 3 & 3\\ -4 & 0 & -6\\ 5 & -1 & 8\end{bmatrix}$$

a) What can you conclude about rank(A) from the information given above?
Since A has a non-zero determinant, we can conclude all columns/rows of A are linearly independent. Thus, A is full rank. For a 2x2 matrix, that means $\boxed{rank(A) = 2}$.

b) Calculate the determinant of B using cofactor expansion across the second column.
Determine the cofactors and write out the three 2x2 determinants for cofactor expansion:

$$\det(B) = -3\begin{vmatrix}-4 & -6\\ 5 & 8\end{vmatrix} + 0\begin{vmatrix}-2 & 3\\ 5 & 8\end{vmatrix} - (-1)\begin{vmatrix}-2 & 3\\ -4 & -6\end{vmatrix}$$

$$= -3((-4*8) - (5*-6)) + 0 + ((-2*-6) - (3*-4)) = \boxed{30}$$

7. For the following statements, determine if they are true or false and why.

a) There exists a square matrix with no eigenvectors.
$\boxed{True}$, for example, $A = \begin{bmatrix}1 & -3\\ 1 & 1\end{bmatrix}$, the characteristic polynomial is $det(A - tI_3) = t^2 - 2t + 4$, which has no real roots and therefore no real eigenvalues.

b) If a matrix A has a eigenvector **v**, then it has infinitely many eigenvectors.
$\boxed{True}$, say **v** is an eigenvector, then any multiple of it is also an eigenvector.

c) The **0** vector is a possible eigenvector.
$\boxed{False}$, if $\mathbf{v} = 0$, then for $A\mathbf{v} = \lambda\mathbf{v}$, λ can be any number, which implies there are infinitely many eigenvalues. This is a moot point, as it does not give us any useful information about the matrix.

8. Given a set of 5 data points that fits the function $\boldsymbol{y} = \beta_1 + \beta_2\boldsymbol{x} - \beta_3\boldsymbol{x}^2 + \beta_4 tan(\boldsymbol{x})$, what would the design matrix for a linear regression look like?
Each column of the design matrix should correspond to one term in the function that is being fit to. Each row corresponds to one of the given data points.

$$X = \begin{bmatrix}1 & x_1 & x_1^2 & tan(x_1)\\ 1 & x_2 & x_2^2 & tan(x_2)\\ 1 & x_3 & x_3^2 & tan(x_3)\\ 1 & x_4 & x_4^2 & tan(x_4)\\ 1 & x_5 & x_5^2 & tan(x_5)\end{bmatrix}$$